Title: A syntactical characterization of structural completeness for implicational logics

Author: Piotr Wojtylak

Bulletin of the **S**ection of **L**ogic
Volume 19/1 (1990), pp. 2–8
reedition 2005 [original edition, pp. 2–9]

Piotr Wojtylak

A SYNTACTICAL CHARACTERIZATION OF STRUCTURAL COMPLETENESS FOR IMPLICATIONAL LOGICS

Let $P, Q, Q_0, Q_1, \ldots$ be propositional formulae in $\{\rightarrow\}$, i.e. they are formulae built up from the propositional variables $p_0, p_1, \ldots$ by use of the operator $\rightarrow$. The structural consequence operation determined in $\{\rightarrow\}$ by (the fragment) of intuitionistic logic is denoted by C_H and the one determined by intuitionistic logic in $\{\rightarrow, \wedge, \vee, \neg\}$ is denoted by C_I.

We examine the problem of structural completeness, with respect to arbitrary finitary and infinitary rules, of any structural consequence operation C in $\{\rightarrow\}$ such that $C \geq C_H$. We prove that the structurally complete extension of C is an extension of C with a certain family of schematically defined rules; the same rules are used for each C. The cardinality of the family is continuum and the family cannot be reduced to a countable one. It means that the structurally complete extension of C_I is not countably axiomatizable by schematic rules. The paper settles a question raised by Professor Wolfgang Rautenberg in [4] which provided an initial stimulus for the present work.

The operation C is structurally complete (see Pogorzelski [2]) if all structural and permissible for C rules are derivable on ground of C. The largest consequence operation among structural C_i's with $C_i(\emptyset) = C(\emptyset)$ is denoted by C^σ. The operation C^σ exists for each C and is structurally complete, see Makinson [1]; C^σ is the operation determined by the structural rules permissible for C. Moreover, there is only one structurally complete operation in the considered family of operations. Thus, C^σ can be said to be the structurally complete extension of C.

The matrix consequence operation determined by the matrix M is denoted by C_M. The matrix M is normal if the *modus ponens* rule is normal in M, i.e. b is distinguished in M whenever $a \to b$ and a are distinguished. The operation C is SFA (strongly finite approximable) if C is the intersection of the operations determined by finite matrices.

Let us consider the infinitary rule ρ determined by the sequent:

$$\frac{\{(p_i \to p_j) \to ((p_j \to p_i) \to p_0) : \text{ for all } 0 < i < j\}}{p_0}$$

Obviously, ρ is a rule of each finite normal matrix in which $p \to p$ is valid and hence ρ is permissible for C_H. On the other hand, ρ cannot be a derivable rule for C_H as p_0 cannot be deduced from any finite subset of $\{(p_i \to p_n) \to ((p_n \to p_i) \to p_0)$: for all $0 < i < j\}$. Thus, see Prucnal [3]:

THEOREM 1. *The operation C_H is not structurally complete.*

A general semantical characterization of structurally complete intermediate logics in $\{\to\}$ was given by Rautenberg in [4], he proved

THEOREM 2. *A structural consequence operation $C \geq C_H$ is structurally complete if and only if C is SFA.*

Rautenberg also asked if there is any syntactical characterization of structurally complete logics. We define here a family of sequential rules and prove that extending with any intermediate logic C one gets C^σ. Hence derivability of rules is a necessary and sufficient condition for structural completeness.

Let Σ be the family of all number theoretic functions f such that $n < f(n)$ for every n, and let r_f, for any $f \in \Sigma$, be an inferential rule in $\{\to\}$ determined by the sequent

$$(*) \quad \frac{\{[\bigwedge_{n<j\leq f(n)} \bigvee_{0<i<j} p_i \equiv p_j] \to p_0 : \text{ for all } n \geq 1\}}{p_0}$$

More specifically, r_f is a rule in $\{\to\}$ which is equivalent in intuitionistic logic to the one defined by $(*)$. Formally,

$$(**) \quad [\bigwedge_{n<j\leq f(n)} \bigvee_{0<i<j} p_i \equiv p_j] \to p_0$$

is not a formula in $\{\to\}$. However, using some intutionistic tautologies one could easily prove that the formulae with $\wedge$ and $\vee$ are equivalent to conjunctions of some formulae in $\{\to\}$. Suppose that $(**)$ is equivalent to a conjunction of some formulae in $\{\to\}$. Suppose that $(**)$ is equivalent to a conjunction of Q^{nk} for $k \leq k_n$. Then r_f is the rule:

$$(***) \quad \frac{\{Q^{nk}: \quad \text{for all} \quad n \geq 1 \quad \text{and all} \quad k \leq k_n\}}{p_0}$$

The exact form of the rule is not important. Let r_f be any rule in $\{\to\}$ equivalent in intuitionistic logic to the one determined by $(***)$.

EXAMPLE 1. Let $f(x) = x + 1$. We get the following rule

$$\frac{\{[\bigwedge_{n<j\leq n+1} \bigvee_{0<i<j} Q_i \equiv Q_j] \to Q_0: \quad \text{for all} \quad n \geq 1\}}{Q_0}$$

which is equivalent to

$$\rho: \quad \frac{\{(Q_i \to Q_j) \to [(Q_j \to Q_i) \to Q_0]: \quad \text{for all} \quad 0 < i < j\}}{Q_0}$$

Thus, one might say that ρ is one of the r_f rules.

EXAMPLE 2. Let us consider the function $f(x) = x + 2$. Let $Z_{ij} := (Q_i \to Q_{n+1}) \to ((Q_{n+1} \to Q_i) \to ((Q_j \to Q_{n+2}) \to ((Q_{n+2} \to Q_j) \to Q_0)))$. Then

$$\frac{\{[\bigwedge_{n<j\leq n+2} \bigvee_{0<i<j} Q_i \equiv Q_j] \to Q_0: \quad \text{for all} \quad n \geq 1\}}{Q_0}$$

is equivalent to

$$\frac{\{Z_{ij}: \quad \text{for all} \quad 0 < i < j < n\}}{Q_0}$$

The above is a rule of any structurally complete intermediate logic in $\{\to\}$ and is not derivable on the ground of the extension of C_H with ρ. It means, in particular, that the extension is not structurally complete.

THEOREM 3. *The rules r_f, for all $f \in \Sigma$, are derivable on the ground of each structurally complete logic C.*

PROOF. Suppose that $f \in \Sigma$ and let M be a finite matrix in which $p_0 \rightarrow p_0$ is valid. Let v be a valuation in M. Since M is finite there is a natural number n_0 such that:

$$\{v(p_i) : 0 < i \leq n_0\} = \{v(p_i) : \quad \text{for all} \quad i \geq 0\}.$$

Then, for every $j > n_0$, there is a number $i < j$ such that $v(p_i) = v(p_j)$. This implies validity of $\wedge_{n<j\leq f(n)} \vee_{0<i\leq j} p_i \equiv p_j$ for every $n \geq n_0$, and hence

$$p_0 \in C_H\{[\wedge_{n<j\leq f(n)} \ \vee_{0<i<j} p_i \equiv p_j] \rightarrow p_0 : \quad \text{for all} \quad n \geq n_0\}$$

as M is normal. Since the above holds for each finite M with $C_M \geq C_H$, it extends to each SFA logic $\geq C_H$. Thus, use of Theorem 2, according to which structurally complete logics are SFA, completes our proof.

LEMMA 1.

$$Q \in C_H(X) \equiv Q \in C_I(X), \quad \textit{for all formulae} \quad Q, X \quad \textit{in} \quad \{\rightarrow\}$$

LEMMA 2.

$$C_I(X, P \vee Q) = C_I(X, P) \cap C_I(X, Q),$$

for all X, P, Q *in* $\{\rightarrow, \wedge, \vee, \neg\}$.

Let C^{Σ} be the extension of C with the rules r_f for all $f \in \Sigma$. Thus, C^{Σ} is the smallest logic $\geq C$ for which the rules r_f are derivable. We have proved above that the rules are derivable for each structurally complete logic. From this it follows that the rules r_f are permissible for each intermediate logic C. So, $C(\emptyset) = C^{\Sigma}(\emptyset)$ and hence one could say that the addition of the rules does not change the set of tautologies of the logic C. Since the structurally complete extension of C, that is the operation C^{σ}, is the greatest structural consequence operation for which $C(\emptyset) = C^{\sigma}(\emptyset)$, we conclude that $C^{\Sigma} \leq C^{\sigma}$. The main result of paper says that the converse also holds, that is

THEOREM 4. *The operation C^{Σ} is the structurally complete extension of C, i.e. $C^{\Sigma} = C^{\sigma}$.*

Proof. It is clear that $C^{\Sigma} \leq C^{\sigma}$. So let us assume that $Q \in C^{\sigma}(X)$ and prove $Q \in C^{\Sigma}(X)$. If $Q \in C(X)$ then obviously $Q \in C^{\Sigma}(X)$; one can assume therefore that $Q \notin C(X)$. Suppose that the sequence $Q_1, Q_2, Q_3, \ldots$ contains all formulae of our language and let us consider the following possibilities:

Case 1. There exists a number n such that for every $m > n$ we have

$$[\bigwedge_{n<j\leq m} \bigvee_{i<j} Q_i \equiv Q_j] \rightarrow Q \notin C(X).$$

Then, according to Lemma 1, we can also have:

$$Q \notin C_I(C(X) \cup \{\bigvee_{i<j} Q_i \equiv Q_j : j > n\}).$$

Let X_0 be a relatively maximal Lindenbaum overset for the formula Q and the set $C(X) \cup \{\bigvee_{i<j} Q_i \equiv Q_j : j > n\}$, that is X_0 is a set such that
(i) $Q \notin C_I(X_0) \supseteq C(X) \cup \{\bigvee_{i<j} Q_i \equiv Q_j : j > n\})$.
(ii) $Q \in C_I(X_0, P)$, for each $P \notin X_0$.
Using above conditions (i) and (ii) and Lemma 2, one can prove that for every $j > n$ there is a number $i < j$ such that $Q_i \equiv Q_j \in X_0$. We conclude, therefore, that Lindenbaum matrix determined by X_0, restricted to the formulae $Q_1, Q_2, Q_3, \ldots$ is finite. Let M denote the matrix. Since $C(\emptyset) \subseteq X_0$ and $C(\emptyset)$ is closed under substitution, all elements of $C(\emptyset)$ are valid in M. Moreover, the matrix M is normal, as X_0 is closed under the *modus ponens* rule, and $Q \notin C_M(X)$. Hence, on the basis of Theorem 2, we get $Q \notin C^{\sigma}(X)$ which means that Case 1 cannot happen at all if $Q \in C^{\sigma}(X)$. Then we get

Case 2. For every n there is a number $m > n$ such that:

$$[\bigwedge_{n<j\leq m} \bigvee_{i<j} Q_i \equiv Q_j] \rightarrow Q \in C(X).$$

Let us take $f(n) = m$ and note that $f \in \Sigma$. Since $C(X) \subseteq C^{\Sigma}(X)$, we obtain $Q \in C^{\Sigma}(X)$ by a single application of the rule r_f.

Let us note that what we have proved above is something more than is claimed in Theorem 4. Namely, apart from $C^{\sigma} = C^{\Sigma}$, we have proved there that each formula $Q \in C^{\Sigma}(X)$ has a relatively simple proof of the ground of the logic C^{Σ}. Thus, though the rules r_f seem to be artificial, they generate quite simple and natural proof system.

THEOREM 5. *For every structural consequence operation $C \geq C_H$ and every formulae X, Q in $\{\to\}$: we have $Q \in C^{\Sigma}(X)$ if and only if*

(i) $Q \in C(X)$, *or*

(ii) Q *can be given by a single application of one of the rules* r_f, *for* $f \in \Sigma$, *with respect to formulae in* $C(X)$.

Moreover, we have the following syntactical characterization of structurally complete intermediate logics in $\{\to\}$.

COROLLARY 5. *A structural consequence operation $C \geq C_H$ is structurally complete if and only if the rules r_f, for all $f \in \Sigma$, are derivable on the ground of C.*

There raises a question if the results proved here preserve their validity when one extends the language $\{\to\}$ by adding other propositional operators. A quick inspection of our argumentation reveals it strongly relies on Theorem 2 which does not old in extended languages. So, the same results as above can be shown for intermediate logics in $\{\to, \wedge\}$, but not in e.g. $\{\to, \vee\}$. Some fragments of our argumentation can be modified, however, and used to prove results for logics in $\{\to, \wedge, \vee, \neg\}$ e.g.

THEOREM 6. *The consequence operation C_I^{Σ}, that is the extension of intuitionistic logic with the rules r_f (defined in the extended language) is SFA.*

The operation C_I^{Σ} is determined by all finite Heyting algebras, i.e.

$$C_I^{\Sigma} = inf\{C_M : M \text{ is a finite Heyting algebra}\},$$

and is not finite. It means that $C_I^{\Sigma} \neq C_I$, though $C_I^{\Sigma}(X) = C_I(X)$ for each finite X. Moreover, C_I^{Σ} is not structurally complete.

It can be easily seen that the cardinality of Σ is continuum, so the structurally complete extension of C is axiomatized by use of uncountably many sequential rules. We can prove that this axiomatization cannot be reduced to a countable one:

THEOREM 7. *The consequence operation C_H^{σ} cannot be axiomatized by use of any countable family of sequential rules.*

References

[1] D. Makinson, *A characterization of structural completeness of a structural consequence operation*, **Reports on Mathematical Logics**, vol. 6 (1976), pp. 99–101.

[2] W. A. Pogorzelski, *Structural completeness of a propositional calculus*, **Bulletin de l'Academie Polonaise des Sciences, serie des Sciences Mathematiques, Astronomiques et Phisiques**, vol. 19 (1971), pp. 349–351.

[3] T. Prucnal, *Structural completeness of a purely implicational intermediate logics*, **Foundations of logic and Linguistic**, (Salzburg 1983), Plenum, London-New York 1985, pp. 31–41.

[4] W. Rautenberg, *A note on implicational intermediate consequence*, **Bulletin of the Section of Logic**, vol. 14 (1985), pp. 103–108.

Institute of Mathematics
Silesian University
Bankowa 14, Katowice 40–007, Poland