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ANTI-CHAINS, FOCUSES AND PROJECTIVE FORMULAS

Abstract

We characterize projective formulas in intuitionistic propositional logic in terms of properties of subsets of universal Kripke models they define. The characterization allows us to prove some properties of the formulas in question.

1. Preliminaries

The notion of projective formula was introduced by S. Ghilardi and, in the series of papers including [2], [3], [4], it was shown by him that projective formulas play a significant role in considerations of unification in propositional logics. In this note we consider projective formulas in intuitionistic propositional logic and, in our central result, we show that they can be naturally characterized in terms of the so-called universal Kripke models. The universal Kripke models for intuitionistic propositional logic correspond to the free finitely generated Heyting algebras and were described by A. Urquhart in [7] and F. Bellissima in [1]. In this note, we introduce the notion of balanced subset of the universal model and show that a formula φ is projective in intuitionistic propositional logic if and only if the subset of the appropriate universal Kripke model which is defined by φ is balanced. Together with the results of F. Bellissima from [1], concerning definable subsets of universal models, the characterization in question enables us to prove some further properties of projective formulas.

Let us begin with quoting basic definitions and facts used later in the paper. We consider intuitionistic propositional logic, IPC, whose formulas are built up from propositional variables by means of propositional connectives $\vee$, $\wedge$, $\rightarrow$ and $\neg$. The symbol P_n will stand for the set of propositional variables $\{p_i : i < n\}$. We use the letters φ, ψ to denote formulas and F_n for the set of all formulas in the variables in P_n. We say that a formula $\varphi \in F_n$ is *projective* (in L) iff there is a substitution $\pi : P_n \rightarrow F_n$ such that $\vdash \pi(\varphi)$ and $\varphi \vdash p \leftrightarrow \pi(p)$, for all $p \in P_n$ (where $\vdash$ stands for the derivability in L).

We briefly recall some general definitions and constructions concerning Kripke semantics for IPC which will be used later.

By an *intuitionistic Kripke model over* P_n, or shortly a P_n*-model*, we mean a tuple $\mathcal{K} = (K, \leq, \Vdash)$ where $(K, \leq)$ is a partially ordered set called a *frame* of $\mathcal{K}$ and $\Vdash \subseteq K \times P_n$ is a *forcing relation* satisfying the monotonicity condition: if $(\mathcal{K}, x) \Vdash p$ and $x \leq y$ then $(\mathcal{K}, y) \Vdash p$, for every $x \in K$ and $p \in P_n$. The forcing relation is extended to the set of all propositional formulas F_n in the usual way; see [6] for details. For a P_n-model $\mathcal{K} = (K, \leq, \Vdash)$ and a node $x \in K$ we will write $V_{\mathcal{K}}(x)$ for $\{p \in P_n : (\mathcal{K}, x) \Vdash p\}$. We say that $\mathcal{K}$ is *rooted* if there is the least element in its frame. Recall that IPC is sound and complete with respect to the class of all finite rooted Kripke models.

In the sequel we will consider some special subsets of the frames of models. Here we quote the appropriate notions. Let $(X, \leq)$ be a partially ordered set. For a subset S of X we set

$$S{\uparrow} = \{x \in X : \text{there exists } s \in S \text{ with } s \leq x\};$$

we call S *upward closed* iff $S = S{\uparrow}$. Similarly,

$$S{\downarrow} = \{x \in X : \text{there exists } s \in S \text{ with } x \leq s\}.$$

We will denote the set of all minimal (maximal) elements of S by $S_{\min}$ (and $S_{\max}$ respectively). Finally, we set

$$S{\uparrow\downarrow\downarrow} = \{x \in X : \{x\}{\uparrow} \subseteq S{\uparrow} \cup S_{\min}{\downarrow}\}.$$

Notice that for each $S \subseteq X$, the sets $S{\uparrow}$, $X \setminus S{\downarrow}$ and $S{\uparrow\downarrow\downarrow}$ are upward closed.

Let A be an anti-chain in $(X, \leq)$. We call an element $x \in X$ a *focus for* A if A is the set of all immediate successors of x in $(X, \leq)$, i.e., if

$A = \big(\{x\}\uparrow \setminus \{x\}\big)_{\min}$. The set of all focuses of an anti-chain A will be denoted by Focus(A).

Finally, recall that every propositional formula $\varphi \in F_n$ defines an upward closed subset of a P_n-model $\mathcal{K} = (K, \leq, \Vdash)$:

$$\mathcal{K}(\varphi) = \{x \in K : (\mathcal{K}, x) \Vdash \varphi\}.$$

The central notion of this note is that of universal Kripke models for IPC. Below we briefly describe a construction of such models. A formal definition and examples can be found in [1].

Let us fix $n \in \omega$. The construction of n-universal model, $\mathcal{U}_n$, can be described by defining the consecutive levels of $\mathcal{U}_n$, where by the k-th level of $\mathcal{U}_n$, denoted by $\mathrm{Lev}_k(\mathcal{U}_n)$, we mean the set of all nodes of the model $\mathcal{U}_n$ whose depth equals k. Note that each level of $\mathcal{U}_n$ consists of a finite number of nodes forming an anti-chain.

The topmost level of $\mathcal{U}_n$, $\mathrm{Lev}_0(\mathcal{U}_n)$, contains 2^n nodes forming an anti-chain and each node of $\mathrm{Lev}_0(\mathcal{U}_n)$ forces a different subset of the set of variables P_n, i.e., for each $Q \subseteq P_n$ we have $x_Q \Vdash_0 p$ iff $p \in Q$. Assume that the levels $\mathrm{Lev}_0(\mathcal{U}_n), \ldots, \mathrm{Lev}_k(\mathcal{U}_n)$ are already constructed. We construct $\mathrm{Lev}_{k+1}(\mathcal{U}_n)$, i.e., the anti-chain of the elements of $\mathcal{U}_n$ of depth $k+1$. Firstly, for each $x \in \mathrm{Lev}_k(\mathcal{U}_n)$ and each proper subset Q of the set $\{p \in P_n : x \Vdash_k p\}$ we add to $\mathrm{Lev}_{k+1}(\mathcal{U}_n)$ a node x_Q such that $x_Q < x$ and set $x_Q \Vdash_{k+1} p$ iff $p \in Q$. Now, let $A = \{x_1, \ldots, x_m\}$ be an anti-chain of which at least one element is in $\mathrm{Lev}_k(\mathcal{U}_n)$. Then for each subset Q of the set $\bigcap_{1 \leq i \leq m}\{p \in P_n : x_i \Vdash_k p\}$ we add to $\mathrm{Lev}_{k+1}(\mathcal{U}_n)$ a node x_Q such that x_Q is a focus of A and set $x_Q \Vdash_{k+1} p$ iff $p \in Q$. We define $\mathcal{U}_n = (U, \leq, \Vdash)$ such that $U = \bigcup_{i \in \omega} \mathrm{Lev}_i(\mathcal{U}_n)$, the relation $\leq$ is the transitive and reflexive closure of $<$ and the forcing relation $\Vdash$ of $\mathcal{U}_n$ is the union of the relations $\Vdash_i$ described above. Let us note that if A is an anti-chain in a universal model $\mathcal{U}_n$, then different focuses for A force different subsets of P_n.

Note that the model $\mathcal{U}_n$ is a P_n-model. Moreover, it can be proved that for every formula φ in F_n, $\mathcal{U}_n \Vdash \varphi$ iff $\vdash_{\mathsf{IPC}} \varphi$. It also well-known that for any finite rooted P_n-model $\mathcal{K}$ there is p-morphism of $\mathcal{K}$ onto some submodel $\mathcal{M}$ of the universal model $\mathcal{U}_n$. In particular, the models $\mathcal{K}$ and $\mathcal{M}$ force exactly the same formulas.

The following two facts were proved by F. Bellissima.

THEOREM 1 ([1]). *For each finite subset X of $\mathcal{U}_n$ the sets $X\uparrow$, $\mathcal{U}_n \setminus X\downarrow$ and $X\uparrow\downarrow\downarrow$ are definable.*

THEOREM 2 ([1]) *Let $\varphi \in F_n$. Then*

1. *φ is dense (i.e., it is a classical tautology which is not provable in* IPC*) iff $\mathcal{U}_n(\varphi) \supseteq \mathrm{Lev}_0(\mathcal{U}_n)$,*
2. *φ is regular (i.e., φ is equivalent in* IPC *to a negated formula) iff $\mathcal{U}_n(\varphi) = X\uparrow\downarrow\downarrow$ for some $X \subseteq \mathrm{Lev}_0(\mathcal{U}_n)$,*
3. *φ is $\wedge$-irreducible (i.e., it is not equivalent in* IPC *to a conjunction $\psi \wedge \psi'$ with ψ and ψ' not equivalent to φ in* IPC*) iff $\mathcal{U}_n(\varphi) = \mathcal{U}_n \setminus \{u\}\downarrow$ for some $u \in \mathcal{U}_n$.*

2. A characterization of projective formulas

In this section we give a characterization of projective formulas of IPC in terms of subsets of universal Kripke models. In the proof we will rely on the characterization of projective formulas provided by S. Ghilardi which refers to the so-called extension property, the notion which we now briefly recall.

Let $\mathcal{K} = (K, \leq, \Vdash)$ and $\mathcal{K}' = (K, \leq, \Vdash')$ be two rooted Kripke P_n-models on the same frame $(K, \leq)$. We say that $\mathcal{K}'$ is a *variant* of $\mathcal{K}$ iff for every $p \in P_n$ and every $x \in K$ except the root, $(\mathcal{K}, x) \Vdash p$ iff $(\mathcal{K}', x) \Vdash' p$. For rooted P_n-models $\mathcal{K}_i = (K_i, \leq_i, \Vdash_i)$, where $1 \leq i \leq m$, we consider the P_n-model $(\sum_i \mathcal{K}_i)'$ whose frame results in joining the disjoint union of the frames $(K_i, \leq_i)$ with a new root, and whose forcing relation is the union of the relations $\Vdash_i$. Note that it follows that the root does not force any propositional variable. We say that a class of finite rooted Kripke models $\mathcal{C}$ has the *extension property* iff for any finite number of models $\mathcal{K}_1, \ldots, \mathcal{K}_m \in \mathcal{C}$ there is a variant of the model $(\sum_i \mathcal{K}_i)'$ which belongs to $\mathcal{C}$.

The following theorem is due to S. Ghilardi.

THEOREM 3 ([2]) *A formula φ is projective iff the class of all finite rooted models of φ has the extension property.*

Before we prove our characterization let us state the following definition.

DEFINITION. Let $(X, \leq)$ be a partially ordered set. We say that a subset S of X is *balanced* if S is upward closed and each finite anti-chain $A \subseteq S$ has a focus in S.

Now we can characterize projective formulas as these which define balanced subsets of appropriate universal models.

THEOREM 4. *Let $\varphi \in F_n$. Then φ is projective in* IPC *iff $\mathcal{U}_n(\varphi)$ is balanced.*

PROOF. Let $\varphi \in F_n$ and suppose that $\mathcal{U}_n(\varphi)$ is not balanced. Then there is an anti-chain A in $\mathcal{U}_n(\varphi)$ such that none of its focuses is in $\mathcal{U}_n(\varphi)$. Suppose that $A = \{x_1, \ldots, x_m\}$ for some $m \geq 2$ and for each $1 \leq i \leq m$ consider the submodel $\mathcal{K}_i$ of $\mathcal{U}_n$ generated by x_i. Since none of the focuses of A is in $\mathcal{U}_n(\varphi)$ it follows that there is no variant $\mathcal{K}$ of the model $(\sum_i \mathcal{K}_i)'$ such that $\mathcal{K} \Vdash \varphi$.

Conversely, assuming that $\mathcal{U}_n(\varphi)$ is balanced we will show that φ has the extension property. Suppose that $\mathcal{K}_1, \ldots, \mathcal{K}_m$ are rooted models of φ with the roots $x_1, \ldots, x_m$ respectively. Then for every $1 \leq i \leq m$ there is $u_i \in U$ such that the submodel $\mathcal{M}_i$ of $\mathcal{U}_n$ generated by u_i is a p-morphic image of $\mathcal{K}_i$. Let

$$X = \{u_i : 1 \leq i \leq m\}{\uparrow}.$$

Then X_{min} is an anti-chain and, since $\mathcal{K}_i \Vdash \varphi$ for every i, $X_{\text{min}} \subseteq \mathcal{U}_n(\varphi)$. By the assumption, $\mathcal{U}_n(\varphi)$ is balanced, so there is a focus u of X_{min} which belongs to $\mathcal{U}_n(\varphi)$. We define a variant $\mathcal{K}$ of the model $(\sum_i \mathcal{K}_i)'$ by putting

$$(\mathcal{K}, x) \Vdash p \iff (\mathcal{U}_n, u) \Vdash p,$$

for all $p \in P_n$, where x is the root of $(\sum_i \mathcal{K}_i)'$. It is easy to see that the submodel of $\mathcal{U}_n$ generated by x is a p-morphic image of the model $\mathcal{K}$, and in particular since $x \in \mathcal{U}_n(\varphi)$, we have $\mathcal{K} \Vdash \varphi$. Thus, the class of rooted models of φ has extension property and hence, by Theorem 3, φ is projective. $\square$

3. Balanced subsets of the form $X\!\uparrow$, for finite X

In this section we consider the balanced subsets of universal models of the form $X\!\uparrow$ where X is finite. Note that, by the properties of the universal models, finiteness of X implies finiteness of $X\!\uparrow$. So, in fact, we will describe the set of all finite balanced subsets of universal models. By Theorem 2, we know that all such sets are definable by formulas of IPC. We draw some consequences of this fact.

Let us start with two lemmas.

LEMMA 5. *Let X be a balanced subset of $\mathcal{U}_n$. Then, if there is a three-element anti-chain in X then X is infinite.*

PROOF. Let $X \subseteq \mathcal{U}_n$ be balanced. Assume that $A = \{x, y, z\}$ is an anti-chain in X and let k be the greatest index such that $\mathrm{Lev}_k(A) \neq \emptyset$. We may assume that $x \in \mathrm{Lev}_k(A)$.

Then, since X is balanced, it follows that $\mathrm{Lev}_{k+1}(X)$ must contain the focuses for the three anti-chains $\{x, y\}$, $\{x, z\}$ and $\{x, y, z\}$. Now, let

$$a_i = |\mathrm{Lev}_{k+1+i}(X)|.$$

We have $a_0 \geq 3$. Moreover, since for each i, the anti-chain $\mathrm{Lev}_{k+2+i}(X)$ must contain a focus for every anti-chain in $\mathrm{Lev}_{k+1+i}(X)$, we have $a_{i+1} \geq \sum_{j=2}^{a_i} \binom{a_i}{j}$. Hence,

$$a_i = \binom{a_i}{a_{i-1}} < \sum_{j=2}^{a_i} \binom{a_i}{j} \leq a_{i+1}.$$

Thus $(a_i)_{i\in\omega}$ is a strictly increasing sequence of positive numbers. It follows, in particular, that X is infinite. □

LEMMA 6. *Let X be a balanced subset of $\mathcal{U}_n$. Then, if $x \in \mathrm{Lev}_i(X)$ and $y \in \mathrm{Lev}_j(X)$ for $i < j$ and $\{x, y\}$ is an anti-chain then there are $x_1 \in \mathrm{Lev}_{i+1}(X)$ and $y_1 \in \mathrm{Lev}_{j+1}(X)$ such that $\{x_1, y_1\}$ is also an anti-chain.*

PROOF. Let x, y satisfy the assumption of the Lemma. Since, in particular, X is upward closed, then there is $z > y$ such that $z \in \mathrm{Lev}_i(X)$ and thus

$\{x, z\}$ is an anti-chain. Since X is balanced, we can find $x_1 \in \text{Focus}(\{x, z\})$ and $y_1 \in \text{Focus}(\{x, y\})$. such that $x_1 \in \text{Lev}_{i+1}(X)$ and $y_1 \in \text{Lev}_{j+1}(X)$. It follows that it is not the case that $x_1 \leq y_1$. Observe that $y_1 < y < z$ and $x_1 < x$. Hence it is also impossible that $y_1 < x_1$ since $y_1 \in \text{Focus}(\{x, y\})$. So, $\{x_1, y_1\}$ is an anti-chain. □

Now we are ready to prove our characterization of finite balanced subsets of universal models.

THEOREM 7. *Let X be a finite subset of $\mathcal{U}_n$. Then $X{\uparrow}$ is balanced iff X is rooted and for every anti-chain A in $X{\uparrow}$, $|A| = 2$ and $A \subseteq \text{Lev}_k(X{\uparrow})$ for some $k < \omega$.*

PROOF. Assume that set X is a finite balanced subset of $\mathcal{U}_n$. It is clear that $X{\uparrow}$ is rooted, since otherwise its minimal elements would form an anti-chain $A \subseteq X{\uparrow}$ such that $\text{Focus}(A) \cap X{\uparrow} = \emptyset$. By Lemma 5 it is clear that for every anti-chain A in $X{\uparrow}$ we have $|A| = 2$. We prove that for every anti-chain $A \subseteq X{\uparrow}$ there is $k < \omega$ such that $A \subseteq \text{Lev}_k(X{\uparrow})$. Suppose that it is not the case, i.e., there are $x \in \text{Lev}_i(X{\uparrow})$ and $y \in \text{Lev}_j(X{\uparrow})$ such that $i < j$ and $\{x, y\}$ is an anti-chain. Then x and y satisfy the assumptions of Lemma 6. Using Lemma 6 in the induction step, we easily show that for every $k \geq j$ the set $\text{Lev}_k(X{\uparrow})$ is non-empty, quod non.

Now assume that X is a finite rooted subset of $\mathcal{U}_n$ and that every anti-chain A in $X{\uparrow}$ has only two elements and lies at some level of $\mathcal{U}_n$. We will prove that $X{\uparrow}$ is balanced.

Let $A = \{u, v\} \subseteq \text{Lev}_k(X{\uparrow})$ be an anti-chain. Since X is rooted, there must be an element x such that $x \in \text{Lev}_{k+1}(X{\uparrow})$. We will show that $x \in \text{Focus}(A)$. Observe that $x < u$ and $x < v$, since otherwise we would find an anti-chain contradicting our assumptions. So, $x \in \text{Focus}(B)$ for some anti-chain $B \subseteq \mathcal{U}_n$ containing A. But then $B \subseteq X{\uparrow}$, hence $B = A$, because A cannot be a subset of any other anti-chain in $X{\uparrow}$. It follows that x is a focus for A, and consequently $X{\uparrow}$ is balanced. □

By Theorem 7 it follows the following observation.

PROPOSITION 8. *If X is a finite balanced subset of $\mathcal{U}_n$ then, for each $k \leq \omega$, $\text{Lev}_{k+1}(X) \subseteq \text{Focus}(\text{Lev}_k(X))$.*

We can complement Theorem 7 by showing that finite balanced subsets of $\mathcal{U}_n$ do not contain to long chains.

THEOREM 9. *If* $X \subseteq \mathcal{U}_n$ *is a finite balanced set then* $\mathrm{Lev}_m(X) = \emptyset$ *for all* $m > n+1$.

PROOF. Let X be a finite balanced subset of $\mathcal{U}_n$. Observe that the number of propositional variables which are forced at the consecutive levels of X is strictly decreasing. Indeed, if $\mathrm{Lev}_k(X)$ consists of one element, then each $y \in \mathrm{Lev}_{k+1}(X)$ is a focus for $\{x\}$ and $V_{\mathcal{U}_n}(y) \subset V_{\mathcal{U}_n}(x)$ by the construction of $\mathcal{U}_n$. If $\mathrm{Lev}_k(X) = \{x_1, x_2\}$, then $V_{\mathcal{U}_n}(x_1) \neq V_{\mathcal{U}_n}(x_2)$ and for each $y \in \mathrm{Lev}_{k+1}(X)$ we have $V_{\mathcal{U}_n}(y) \subseteq V_{\mathcal{U}_n}(x_1) \cap V_{\mathcal{U}_n}(x_2)$. Hence $V_{\mathcal{U}_n}(y) \subset V_{\mathcal{U}_n}(x_1) \cup V_{\mathcal{U}_n}(x_2)$.

So, since the number of propositional variables which are forced at the nodes of $\mathrm{Lev}_0(X)$ is at most n, if $\mathrm{Lev}_n(X)$ is non-empty then no propositional variable is forced at the nodes of $\mathrm{Lev}_n(X)$. Moreover, in this case, $\mathrm{Lev}_{n+1}(X)$ contains only one element x, and since none of the propositional variables can be forced at x, $\mathrm{Lev}_{n+2}(X) = \emptyset$. □

From Theorem 9 follows the following fact concerning finite balanced subsets of $\mathcal{U}_n$.

PROPOSITION 10. *For every* n *there is finitely many finite balanced subsets of* $\mathcal{U}_n$.

¿From the Proposition above it follows that for every n there is only finitely many projective formulas that define finite subsets of $\mathcal{U}_n$. It is worth noting that there is only four projective formulas in one variable (excluding IPC-provable ones) and they define finite subsets of $\mathcal{U}_1$. Among them $\neg\neg p \to p$ is the only one that is dense. It contrasts with the general case.

PROPOSITION 11. *If* $n > 1$ *then every dense projective formula defines an infinite subset of* $\mathcal{U}_n$.

PROOF. Let $n > 1$ and $\varphi \in F_n$ be a dense projective formula. Notice that since φ is dense, $\mathcal{U}_n(\varphi) \supseteq \mathrm{Lev}_0(\mathcal{U}_n)$. On the other hand, since $n \geq 2$, we have $|\mathrm{Lev}_0(\mathcal{U}_n\varphi))| = 2^n > 2$ which means that $\mathcal{U}_n(\varphi)$ contains a three-element anti-chain. Hence, by Lemma 5, $\mathcal{U}_n(\varphi)$ is infinite. □

4. Balanced subsets of the form $X{\uparrow}{\downarrow}{\downarrow}$

We turn to a characterization of balanced subsets of universal models of the form $X{\uparrow}{\downarrow}{\downarrow}$ where X is finite.

THEOREM 12. *Let X be a finite subset of $\mathcal{U}_n$. Then $X{\uparrow}{\downarrow}{\downarrow}$ is balanced iff every finite anti-chain $A \subseteq X{\uparrow} \setminus X_{\min}$ has a focus in $X{\uparrow}$.*

PROOF. Obviously, if $X{\uparrow}{\downarrow}{\downarrow}$ is balanced then, in particular, every finite anti-chain $A \subseteq X{\uparrow} \setminus X_{\min}$ has a focus in $X{\uparrow}$. So, it is enough to prove that the converse is also true. Let A be an arbitrary finite anti-chain in $X{\uparrow}{\downarrow}{\downarrow}$. If $A \subseteq X{\uparrow} \setminus X_{\min}$, we are done. So, let $t \in A$ be an element of $X_{\min}{\downarrow}$ and let u be a focus of A. Since $u \leq t$, we have $u \in X_{\min}{\downarrow}$. Moreover, $\{u\}{\uparrow} \setminus \{u\} \subseteq \bigcup_{x \in A}\{x\}{\uparrow}$. On the other hand, since $A \subseteq X{\uparrow}{\downarrow}{\downarrow}$, for every $x \in A$ we have $\{x\}{\uparrow} \subseteq X{\uparrow} \cup X_{\min}{\downarrow}$. Hence $\{u\}{\uparrow} \subseteq X{\uparrow} \cup X_{\min}{\downarrow}$ and it follows that u is in $X{\uparrow}{\downarrow}{\downarrow}$. So, $X{\uparrow}{\downarrow}{\downarrow}$ is balanced. □

An immediate consequence of Theorem 12 is the following

PROPOSITION 13. *If $X \subseteq \mathrm{Lev}_0(\mathcal{U}_n)$ then the set $X{\uparrow}{\downarrow}{\downarrow}$ is balanced.*

Recall, that by Theorem 2, the formulas that define sets of the form $X{\uparrow}{\downarrow}{\downarrow}$, for $X \subseteq \mathrm{Lev}_0(\mathcal{U}_n)$, are regular. Hence, as a corollary, we get the following fact.

COROLLARY 14. *Every regular formula is projective.*

5. Balanced subsets of the form $\mathcal{U}_n \setminus X{\downarrow}$

In this section we consider subsets of universal models that can be presented in the form $\mathcal{U}_n \setminus X{\downarrow}$ for some finite set X. We begin with a characterization of balanced subset of this kind.

THEOREM 15. *Let X be a finite subset of $\mathcal{U}_n$. Then the set $\mathcal{U}_n \setminus X{\downarrow}$ is balanced iff $X_{\max}$ does not include* Focus(A) *for any finite anti-chain A.*

PROOF. Firstly, observe that we have $X{\downarrow} = X_{\max}{\downarrow}$. Indeed, the inclusion $X_{\max}{\downarrow} \subseteq X{\downarrow}$ is obvious. To show the converse assume that $u \in X{\downarrow}$. Then we will find $x \in X$ such that $u \leq x$. On the other hand the set X (and also $\{x\}{\uparrow}$) is finite, hence there must be a maximal element in

$X \cap \{x\}\uparrow$. Consequently, there is $y \in X_{\max}$ with $u \leq y$ and thus $u \in X_{\max}\downarrow$. Of course, it follows that $\mathcal{U}_n \setminus X\downarrow = \mathcal{U}_n \setminus X_{\max}\downarrow$.

It is clear that if $\mathcal{U}_n \setminus X\downarrow$ is balanced then, in particular, $X_{\max}$ cannot contain Focus(A) for any finite anti-chain $A \subseteq \mathcal{U}_n \setminus X_{\max}\downarrow$. Of course, $X_{\max}$ does not contain Focus(A) of any other finite anti-chain too.

To show the converse, suppose that $\mathcal{U}_n \setminus X_{\max}\downarrow$ is not balanced and let A be a finite anti-chain in $\mathcal{U}_n \setminus X_{\max}\downarrow$ such that Focus$(A) \subseteq X_{\max}\downarrow$. Assume that Focus$(A) \subseteq X_{\max}\downarrow \setminus X_{\max}$. Let $u \in$ Focus(A). Then there is $x \in X_{\max}$ such that $u < x$. If x were incomparable with all $t \in A$, we would get $x \in A$, quod non. Of course, it is not the case that $u < x \leq t$ for any $t \in A$. So, we get $t < x$ for some $t \in A$ which contradicts the assumption that $A \subseteq \mathcal{U}_n \setminus X_{\max}\downarrow$. In consequence, we get Focus$(A) \subseteq X_{\max}$. □

We state some consequences of Theorem 15. Firstly, notice that, obviously, $\mathrm{Lev}_0(\mathcal{U}_n)$ does not contain focuses of any anti-chains. Hence we get the following fact.

PROPOSITION 16. *For every $X \subseteq \mathrm{Lev}_0(\mathcal{U}_n)$ the set $\mathcal{U}_n \setminus X\downarrow$ is balanced.*

The following consequence of Theorem 15 will allow us to prove some facts about projective formulas.

PROPOSITION 17. *Let A be a finite anti-chain in $\mathcal{U}_n$ and let $X \subset$ Focus(A). Then the set $\mathcal{U}_n \setminus X\downarrow$ is balanced.*

PROOF. Notice that, since $X \subset$ Focus(A), the set X cannot include the Focus(B) for any other anti-chain B. □

We can also prove the following fact.

PROPOSITION 18. *If Z is upward closed then the set $\mathcal{U}_n \setminus Z\downarrow$ is balanced.*

PROOF. By Proposition 17 and the fact that when Z is upward closed then $\mathcal{U}_n \setminus Z\downarrow = \mathcal{U}_n \setminus \mathrm{Lev}_0(Z)\downarrow$. □

We noticed that there is only one monadic projective formula which is dense. Now we will prove that this, again, is an exceptional case. Let us start with a lemma.

LEMMA 19. *Let $n > 1$. Then $\mathcal{U}_n$ contains infinitely many anti-chains A such that $|\mathrm{Focus}(A)| > 1$. Moreover, the anti-chains in question can be chosen as two-element subsets of $\mathcal{U}_n$.*

PROOF. Let p and q be different propositional variables. Notice that $\mathcal{U}_n(p)$ is a balanced subset of $\mathcal{U}_n$. Consider the nodes $x, z \in \mathrm{Lev}_0(\mathcal{U}_n(p))$ such that $V_{\mathcal{U}_n}(x) = \{p\}$, and $V_{\mathcal{U}_n}(z) = \{p, q\}$. Let $y \in \mathrm{Focus}(\{z\})$ be such a node that $V_{\mathcal{U}_n}(y) = \{p\}$. Of course, $y \in \mathrm{Lev}_1(\mathcal{U}_n(p))$ and $\{x, y\}$ is an anti-chain. Hence, the elements x, y satisfy the assumption of Lemma 6. Now, by induction on k, we show that there is infinitely many anti-chains in $\mathcal{U}_n(p)$. Moreover, since there are exactly two subsets of the set $\{p\}$, for every such an anti-chain A we have exactly two focuses in $\mathcal{U}_n$. □

COROLLARY 20. *For every $n > 1$, there is infinitely many dense projective formulas in F_n.*

PROOF. Fix $n > 1$. By Lemma 19, there is infinitely many finite anti-chain A such that $\mathrm{Lev}_k(A) \neq \emptyset$ for some $k > 0$ and such that $\mathrm{Focus}(A)$ has at least two elements. For an arbitrary $x \in \mathrm{Focus}(A)$, the set $\mathcal{U}_n \setminus \{x\}\!\downarrow$ is defined by a formula φ that is projective by Theorem 15. We can choose the anti-chains A to be two-element subsets of $\mathcal{U}_n$, so the formula φ is dense, because we can choose x in such a way that $\mathrm{Lev}_0(\mathcal{U}_n) \subseteq \mathcal{U}_n \setminus \{x\}\!\downarrow$. □

Note that there are only three $\wedge$-irreducible projective formulas in one variable: $\neg p$, $\neg\neg p$ and $\neg\neg p \to p$. We conclude with the following consequence of Theorem 15 which shows that this is another peculiarity of the monadic fragment of IPC.

COROLLARY 21. *For $n > 1$, there is infinitely many $\wedge$-irreducible projective formulas in F_n.*

PROOF. By Lemma 19 there is infinitely many anti-chains in $\mathcal{U}_n$ that have at least two focuses. For each such an anti-chain A take $x \in \mathrm{Focus}(A)$. Then the formula which defines $\mathcal{U}_n \setminus \{x\}\!\downarrow$ is $\wedge$-irreducible and, by Theorem 15, projective. □

Finally, let us note that for $n > 1$ there is also infinitely many projective formulas in F_n that are not dense and infinitely many formulas that are $\wedge$-reducible.

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