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REMARKS ON PROJECTIVE UNIFIERS

Abstract

A projective unifier for a unifiable formula α in a logic L is a unifier σ for α (i.e. a substitution making α a theorem of L) such that $\alpha \vdash_L \sigma(x) \leftrightarrow x$. Using the result of Burris [3] we observe that every discriminator variety has projective unifiers. Several examples of projective unifiers both in discriminator and in non-discriminator varieties are given. As an application we show that logics with projective unifiers are almost structurally complete, i.e. every admissible rule with unifiable premises is derivable.

Keywords and phrases: unification, unifiers, projective unifiers, structural completeness.

1. Introduction

Unification and E-unification of terms is widely used in Computer Science, in particular it is fundamental in Automated Deduction and Term Rewriting Systems (see e.g. [2]).

Given an equational theory E and two terms t_1, t_2 (i.e. a "unification problem") a substitution σ is called a *unifier for* t_1, t_2 in E, if $\vdash_E \sigma(t_1) \approx \sigma(t_2)$. The terms t_1 and t_2 are *unifiable* if there is a unifier for them. A substitution σ is *more general* than a substitution τ, $\tau \preceq \sigma$, if there is a substitution θ such that $\vdash_E \theta \circ \sigma \approx \tau$; the relation $\preceq$ is reflexive and transitive.

A most general unifier, a *mgu*, for t_1, t_2, is a unifier that is more general than any unifier for t_1, t_2. It can be interpreted as "the best" solution to the equation $t_1 \approx t_2$. An equational theory E has *unitary unification* if for every two unifiable terms t_1 and t_2, there is a mgu σ such that $\vdash_E \sigma(t_1) \approx \sigma(t_2)$. Unification types can be unitary, finitary, infinitary or nullary depending on number of maximal w.r.t. $\preceq$ unifiers, cf [2],[9].

Instead of equational theory E one considers, equivalently, a corresponding equational class $\mathcal{V}$ of algebras (a variety). Boolean algebras have unitary unification.

Unification theory for equational theories, or for varieties, is translated to the corresponding logics, cf. [9]-[12]. Roughly speaking $\vdash_E t_1 = t_2$ is translated into $\vdash_{E'} (A_1 \to A_2) \wedge (A_2 \to A_1)$, where a formula A_i is obtained from a term t_i by replacing operations with corresponding logical connectives and E' is the logic corresponding to the equational theory E. Hence a unification problem can be reduced to a single formula α and a *unifier for a formula* α in a logic L is a substitution σ such that $\vdash_L \sigma(\alpha)$. A *formula* α *is unifiable* in L, if such σ exists. If τ, σ are substitutions, then $\tau \preceq \sigma$, if there is a substitution θ such that $\vdash_L \theta(\sigma(x)) \leftrightarrow \tau(x)$.

Classical propositional logic CL has unitary unification. Every formula α, unifiable (= consistent) in CL, has a mgu, i.e. a substitution σ such that $\vdash_{CL} \sigma(\alpha)$ and that every unifier τ for α is a special case of σ, i.e. $\vdash_{CL} \theta(\sigma(x)) \leftrightarrow \tau(x)$, for some θ. But this is not the case for intuitionistic logic INT and for some modal logics; S. Ghilardi [10], [11] showed that INT, $K4$, $S4$, $S4Grz$ has finitary (but not unitary) unification,

Projective formulas, formulas corresponding in logic to finitely presented projective algebras, cf.[9], are used by Ghilardi as a key notion in [10], [11]. A formula α is *projective* in a logic L if there is a unifier σ for α in L such that for each $x \in \mathrm{Var}(\alpha)$,

$$\alpha \vdash_L \sigma(x) \leftrightarrow x.$$

As stated in [1], 2.3, "α is a projective formula" translates into logic the fact that the free algebra over $Var(\alpha)$ divided by the congruence generated by the equation "$\alpha = \top$" is projective. Such a unifier σ is now called a *projective unifier* for α, see [1].

For modal logic L over K4, a formula $\Box^+\alpha$ is *projective* in L [1], if there is a unifier σ for $\Box^+\alpha$ such that for each $x \in \mathrm{Var}(\alpha)$,

$$\Box^+\alpha \vdash_L \sigma(x) \leftrightarrow x.$$

Earlier A. Wroński in [15] considered transparent unifiers in intermediate logics which are quite close to projective unifiers. A unifier σ is a

[1]See S.Ghilardi [11], here $\Box^+\alpha$ denotes $\Box\alpha \wedge \alpha$.

transparent unifier for α in L if it is a unifier for α such that for any unifier τ for α: $\vdash_L \tau(\sigma(x)) \leftrightarrow \tau(x)$ for every variable x.

In [5] transparent (in fact projective) unifiers are given for $S5$.

Every projective unifier is a most general unifier but not vice versa. Projective unifiers have many advantages over other most general unifiers. They are preserved under extensions of a logic. They can help in recognizing admissible rules. If every unifiable formula is projective in a logic L then L is (almost) structurally complete, that is, every admissible rule (with unifiable premises) is derivable in L. For the notion of structural completeness see [13].

By going from algebra to logic and back we look at projective unifiers and their applications. In section 2 we show, by a modification of the proof of S. Burris [3], that discriminator varieties have projective unifiers given explicitly by a formula. We give examples of projective unifiers.

Section 3 contains a short survey of results on projective unifiers in: classical logic, some intermediate logics and some modal logics. The examples show that projective unifiers given explicitly by one formula may also appear in non-discriminator varieties. Projective unifiers are applied to structural completeness and to the problem of admissible rules.

1. Projective unifiers in discriminator varieties

S. Burris [3] showed that unification is unitary in (theories determined by) discriminator varieties. We slightly modify his argument to show that discriminator varieties have projective unifiers.

In algebraic considerations we follow the notation and notions of [4], [3], e.g. a subdirectly irreducible algebra and a simple algebra (i.e. one which has only two congruences) in a variety $\mathcal{V}$. The classes of those algebras will be denoted by $\mathcal{V}_{SI}$ and $\mathcal{V}_S$, respectively. We recall some basic facts that are needed. $\mathcal{V}(K)$ denotes a variety generated by a class K, $\mathcal{V}(K) = HSP(K)$.

For a given algebra $\mathfrak{A}$, a term $t(x, y, z)$ is a *discriminator term* for $\mathfrak{A}$ if, for $a, b, c \in A$,

$$t(a, b, c) = \begin{cases} c, & \text{if} \quad a = b, \\ a, & \text{if} \quad a \neq b. \end{cases}$$

A term $s(x, y, z, v)$ is a *switching term* for $\mathfrak{A}$ if, for $a, b, c, d \in A$,

$$s(a, b, c, d) = \begin{cases} c, & \text{if} \quad a = b, \\ d, & \text{if} \quad a \neq b. \end{cases}$$

From a discriminator term one can define a switching term and vice versa.

A variety $\mathcal{V}$ is a *discriminator variety* if there is a class K of algebras which generates $\mathcal{V}$ such that there is a term $t(x, y, z)$ which is a discriminator term for every algebra from K. In fact $t(x, y, z)$ is a discriminator term for the simple algebras of $\mathcal{V}$, i.e. $K = \mathcal{V}_{SI}$. If $\mathcal{V}_{SI} = \mathcal{V}_S$, then $\mathcal{V}$ is called *semisimple.* $\mathcal{F}_{\mathcal{V}}(\omega)$ ($\mathcal{F}_{\mathcal{V}}(n)$) denotes ($n$-generated) free algebra of $\mathcal{V}$.

Examples of discriminator varieties:

Boolean algebras, Boolean rings, rings with $x^n = x$, n-valued Post algebras, monadic algebras, $\mathrm{MV_n}$-algebras, cylindric algebras of dimension n, relation algebras.

Let $\mathfrak{A}$ be an algebra and θ a congruence on $\mathfrak{A}$. Then the following facts are known (see e.g. [3]):

(i) $\mathfrak{A}/\theta$ is a nontrivial simple algebra iff θ is a maximal congruence.

(ii) If $\mathfrak{A}$ is in a semisimple variety and $a, b \in A$, then
$a = b$ iff $a/\theta = b/\theta$.

(iii) Discriminator varieties are semisimple.

Unification in discriminator varieties.

Let $\mathcal{V}$ be a variety. Given two terms $p(x_1, \ldots, x_n)$, $q(x_1, \ldots, x_n)$, a substitution τ, $\tau(x_i) = t_i$, for all $i \leq n$, is called a *unifier* of p and q in $\mathcal{V}$ if the equation $p(t_1, \ldots, t_n) \approx q(t_1, \ldots, t_n)$ holds in $\mathcal{V}$, i.e.
$\models_{\mathcal{V}} p(t_1, \ldots, t_n) \approx q(t_1, \ldots, t_n)$.

If such τ exists, then the terms $p(x_1, \ldots, x_n)$, $q(x_1, \ldots, x_n)$ are *unifiable* in $\mathcal{V}$. Given two unifiers τ and σ, σ is *more general* than τ if there is a substitution ε such that $\models_{\mathcal{V}} \varepsilon \circ \sigma \approx \tau$; the relation $\preceq$ is reflexive and transitive.

For equations $p_i(x_1, \ldots, x_n) \approx q_i(x_1, \ldots, x_n), i = 1, 2$, the relation of semantic entailment $\models_{\mathcal{V}}$ determined by $\mathcal{V}$ is defined as follows:

$p_1(x_1, \ldots, x_n) \approx q_1(x_1, \ldots, x_n) \models_{\mathcal{V}} p_2(x_1, \ldots, x_n) \approx q_2(x_1, \ldots, x_n)$ iff for any $\mathfrak{A} \in \mathcal{V}$ and any $a_1, \ldots, a_n \in A$, whenever $p_1(a_1, \ldots, a_n) = q_1(a_1, \ldots, a_n)$ is true in $\mathfrak{A}$, then $p_2(a_1, \ldots, a_n) = q_2(a_1, \ldots, a_n)$ is true in $\mathfrak{A}$. If $\mathcal{V}$ is semisimple, then it is enough to take $\mathfrak{A} \in \mathcal{V}_S$.

A unifier ε for $p = p(x_1, \ldots, x_n)$ and $q = q(x_1, \ldots, x_n)$ is called *projective* in $\mathcal{V}$ if

$$p \approx q \models_{\mathcal{V}} \varepsilon(x_i) \approx x_i, \text{ for all } i \leq n,$$

or, equivalently, $\models_{\mathcal{V}} p \approx q \rightarrow \varepsilon(x_i) \approx x_i$, for all $i \leq n$.

We will say that a variety $\mathcal{V}$ (or a logic L) *have projective unifiers* if for every unifiable terms (formula) a projective unifier exists.

Corollary 1. *Projective unifiers are preserved under taking subvarieties or extending theories (logics).*

Note that unitary unification is not preserved under taking subvarieties or extending theories (logics), see e.g. [12].

Now we observe that the proof of the theorem of S.Burris [3], stating that discriminator varieties $\mathcal{V}$ have unitary unification, can be modified to prove that $\mathcal{V}$ has projective unifiers.

Theorem 2. *Discriminator varieties have projective unifiers. More exactly, for every dicscriminator variety $\mathcal{V}$ with a switching term $s(x, y, z, v)$ for algebras from $\mathcal{V}_{SI}$, two terms $p = p(x_1, \ldots, x_n)$ and $q = q(x_1, \ldots, x_n)$ unifiable in $\mathcal{V}$, let $r_1, \ldots, r_n$ be terms such that $\models_{\mathcal{V}} p(t_1, \ldots, t_n) \approx q(t_1, \ldots, t_n)$. Then*

$$\sigma(x_i) = s(p, q, x_i, r_i), \text{ for all } i \leq n.$$

is a projective unifier for p and q in $\mathcal{V}$.

Proof: The proof that σ is a unifier for p and q in $\mathcal{V}$ is the same as in [3]. The idea is that for a maximal congruence θ of $\mathcal{F}_{\mathcal{V}}(\omega)$, $\sigma(p)/\theta = \sigma(q)/\theta$ on the simple algebra $\mathcal{F}_{\mathcal{V}}/\theta$. Hence, by the facts (ii),(iii), σ is a unifier for p and q in $\mathcal{V}$. To show that

$$p \approx q \models_{\mathcal{V}} \sigma(x_i) \approx x_i, \text{ for all } i \leq n$$

assume, for $\mathfrak{A} \in \mathcal{V}_S$ and $a_1, \ldots, a_n \in A$, that $p(a_1, \ldots, a_n) = q(a_1, \ldots, a_n)$ holds in $\mathfrak{A}$. Then, by the definition of s,

$\sigma(x_i) = s(p, q, x_i, a_i) = x_i$, for all $i \leq n$, holds in $\mathfrak{A}$. □

As a corollary we provide some examples, in algebra, of projective unifiers which depend on some *ground* unifiers, i.e. unifiers for p and q of the form $\tau_0 : Var\{p, q\} \to \{0, 1\}$ (the 2-element Boolean algebra).

Corollary 3. *Let p and q be unifiable terms and $Var\{p, q\} = \{x_1, \ldots, x_n\}$.*

1. **Boolean algebras** $(B, \wedge, \vee, ', 0, 1)$. *Let $\tau_0 : Var\{p, q\} \to \{0, 1\}$ be a ground unifier for p and q, and $p + q = (p \wedge q') \vee (p' \wedge q)$. Then a projective unifier for p and q has the following form:*

$$\sigma(x_i) = ((p + q)' \wedge x_i) \vee ((p + q) \wedge \tau_0(x_i)), \text{ for all } i \leq n.$$

2. **Monadic algebras** $(B, \wedge, \vee, ', \Diamond, 0, 1)$, *where $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra and $\Diamond$ satisfies the following axioms: $\Diamond 0 = 0$, $\Diamond(x \vee y) =$*

$\Diamond x \vee \Diamond y$, $x \leq \Diamond x$, $\Diamond\Diamond x = \Diamond x$ (i.e. a topological closure algebra) and $\Diamond(\Diamond x)' = (\Diamond x)'$.
Let $\tau_0 : Var\{p, q\} \to \{0, 1\}$ be a ground unifier for p and q, and $p + q = (p \wedge q') \vee (p' \wedge q)$. Then a projective unifier for p and q has the following form:

$$\sigma(x_i) = \big((\Diamond(p+q))' \wedge x_i\big) \vee (\Diamond(p+q) \wedge \tau_0(x_i)), \text{ for all } i \leq n.$$

REMARK. The *dual discriminator term* on an algebra $\mathfrak{A}$, is a ternary term q such that, for $a, b, c \in A$: $q(a, b, c) = c$, if $a \neq b$, and, $q(a, b, c) = a$, if $a = b$. Although discriminator varieties enjoy unitary unification the variety of distributive lattices, which is a dual discriminator variety, has nullary ("the worst") unification (Willard 1991).

There are varieties which enjoy unitary unification but such that projective unifiers do not exist for some unifiable terms; for example, the variety of KC-algebras (or De Morgan algebras) i.e. Heyting algebras satisfying additionally the weak law of excluded middle $\neg\neg x \vee \neg x = 1$ is such, see the logic KC in the next section.

2. Projective unifiers in some logics.

Now we provide some examples of logics with projective unifiers, together with an explicit form of the projective unifiers. The list include examples 1, 2, 3 of logics corresponding to discriminator varieties as well as examples 4, 5, 6, 7 corresponding to which classes of algebras are not discriminator varieties. A *ground* unifier is a unifier of the type : $Form \to \{\bot, \top\}$ (constant falsity and truth, respectively). In examples 1, 2, 3, 4 and 5 a ground unifier is applied in the definitions of projective unifiers. [2]

Examples: For a given logic L in the appropriate language let α be a unifiable formula in L with a ground unifier τ_0. Then a projective unifier ε for α in L has the following form, for $x \in Var(\alpha)$:

1. Classical logic, in $\{\wedge, \vee, \to, \neg\}$,

$$\varepsilon(x) = (\alpha \to x) \wedge (\alpha \vee \tau_0(x)),$$

[2] This is not the case in general. For instance, in intuitionistic logic only some formulas, e.g. $\neg\neg x \to x$ and $B \to x$ have projective unifiers $\varepsilon(x) = \neg\neg x$, $\varepsilon(x) = (B \to x) \to x$, respectively, but $\neg x \vee x$ does not have a mgu, see [10].

2. Modal logic S5 = KT4B, in $\{\wedge, \vee, \rightarrow, \neg, \Box\}$, see [5],
$$\varepsilon(x) = (\Box\alpha \rightarrow x) \wedge (\Box\alpha \vee \tau_0(x)),$$
3. Modal logic $KD4^{[n]}B^{[k]}$, in $\{\wedge, \vee, \rightarrow, \neg, \Box\}$ for $n, k \geq 1$, see [6], [7],
$$\varepsilon(x) = (\Box^n\alpha \rightarrow x) \wedge (\Box^n\alpha \vee \tau_0(x)),$$
4. Logic $INT^{\{\rightarrow\}}$ of intuitionistic implication, in $\{\rightarrow\}$ (and $INT^{\{\rightarrow,\wedge\}}$ of intuitionistic implication and conjunction in $\{\rightarrow, \wedge\}$), based on T. Prucnal [14],
$$\varepsilon(x) = (\alpha \rightarrow x),$$
5. Gödel-Dummett logic LC = INT + $(A \rightarrow B) \vee (B \rightarrow A)$, in $\{\wedge, \vee, \rightarrow, \neg\}$, based on A. Wroński [15],
$$\varepsilon(x) = (\alpha \rightarrow x) \wedge (\neg\neg\alpha \vee \tau_0(x)),$$
6. Logic $INT^{\{\leftrightarrow\}}$ of Intuitionistic Equivalence in $\{\leftrightarrow\}$, see A.Wroński [16]; a form of the unifier is not easy to write.
7. Modal logic S4.3 in $\{\wedge, \vee, \rightarrow, \neg, \Box\}$, the unifiers can not be simply described as depending on a ground unifier (as in 1,2,3,5), see [8].

Examples 4, 5, 6 and 7 above present logics with projective unifiers such that their corresponding varieties are *not* discriminator varieties. For instance in 5, the variety of Gödel algebras is not semisimple.

The logic of weak excluded middle or De Morgan logic, KC, provides an example of a logic in which every unifiable formula has a mgu but some formulas are not projective (projective unifiers do not exist).

S.Ghilardi has shown that KC admits unitary unification [10].

Corollary 4. *The logic $KC = INT + \neg\alpha \vee \neg\neg\alpha$ admits unitary unification but projective unifiers do not exist for some unifiable formulas.*

Proof: Assume, to the contrary, that every unifiable formula has a projective unifier in KC. Consider the formulas $\alpha := (x \wedge y \rightarrow z) \rightarrow x \vee y$ and $\beta := (z \rightarrow x) \vee (z \rightarrow y)$. α is unifiable in KC. Let σ be a projective unifier for α in KC. Since $\alpha \vdash_{KC} \sigma(x) \leftrightarrow x$, for $x \in Var(\alpha)$, can be extended to $\alpha \vdash_{KC} \sigma(\gamma) \leftrightarrow \gamma$, for any γ, we have, in particular $\alpha \vdash_{KC} \sigma(\beta) \leftrightarrow \beta$.

It can be shown that the rule α/β is admissible in KC [3], that is, for every substitution τ, $\vdash_{KC} \tau(\alpha) \Rightarrow\ \vdash_{KC} \tau(\beta)$. Since σ is a unifier for α we

[3]See [7] for comments; professor A. Wroński gave a proof based on Kripke semantics (a private communication).

get $\vdash_{KC} \sigma(\beta)$. From the above, by Modus Ponens, we have $\alpha \vdash_{KC} \beta$, hence $\vdash_{KC} \alpha \to \beta$, but this is false, i.e. α can not have a projective unifier. □

A formula $\alpha = (x \to y) \to x \vee z$ is another non-projective one in KC. On the other hand, all so called Harrop formulas are projective, see [7].

Corollary 5. *The variety KC corresponding to INT + $\neg\alpha\vee\neg\neg\alpha$ contains non-trivial finitely presented algebras which are not projective.*

Proof: Such an algebra is given by the quotient $\mathcal{F}_{KC}(x, y, z)/_{\alpha}$, i.e. the free KC-algebra $\mathcal{F}_{KC}(x, y, z)$, divided by the congruence generated by $\alpha := (x \wedge y \to z) \to x \vee y$. □

3. Applications to Structural Completeness

Let $U(\alpha)$ be a set of all unifiers for a formula α in a logic L and $U_{max}(\alpha)$ be a set of all maximal w.r.t. $\preceq$ elements from $U(\alpha)$. We consider only structural (i.e. preserving substitutions) rules of inference.

A logic L is *almost structurally complete*, if is every admissible rule with unifiable premises, i.e $r : \alpha/\beta$ with $U(\alpha) \neq \emptyset$, is derivable in L. If every consistent formula is unifiable in L then “almost structurally complete” and “structurally complete” (cf. [13]) coincide.

Corollary 6. *If a logic has projective unifiers then it is almost structurally complete.*

This holds for logics determined by discriminator varieties. In Examples (section 2): (1) classical logic, (4) logic $INT^{\{\to\}}$ and $INT^{\{\to,\wedge\}}$ (5) Gödel-Dummett logic LC are structurally complete; on the other hand, modal logics (2) S5, (3) $KD4^{[n]}B^{[k]}$ and (7) S4.3 are almost structurally complete but not structurally complete (as some consistent formulas are not unifiable).

Corollary 7. *The following conditions are equivalent:*

(i) a rule with the schema $r : \alpha/\beta$ is admissible in L,

(ii) $\vdash_L \tau\beta$, for $\tau \in U(\alpha)$, or for $\tau \in U_{max}(\alpha)$.
If unification in L is unitary then (i) iff

(iii) $\vdash_L \tau\beta$, for a mgu τ for α.

Hence one may check that the following rules are admissible in INT:
- $\neg x \to y \vee z \ / \ (\neg x \to y) \vee (\neg x \to z)$ the Kreisel-Putnam rule, hint: use two unifiers: $\sigma_1(y) = (\neg x \to y) \to y$ and $\sigma_2(z) = (\neg x \to z) \to z$;
- $((\neg\neg x \to x) \to (x \vee \neg x)) \ / \ (\neg\neg x \vee \neg x)$ the Scott rule.

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