PRACE NAUKOWE UNIWERSYTETU ŚLĄSKIEGO W KATOWICACH NR 12

PRACE MATEMATYCZNE II, 1972

MAREK ROCHOWSKI

Immersions of two-manifolds in the Euclidean four-space

In this paper the author will investigate the immersions of closed orientable two-manifolds in the Euclidean four-space for which the Gauss curvature of the metric induced by the immersion is not everywhere negative. Hence such immersions cannot be isometries for orientable two-manifolds of genus $\geqslant 2$ regarded as spaces locally isometric with the Lobachevskian plane. The method used is that developed in [1].

1. PRELIMINARIES. Let E^{n+N} denote the $(n+N)$-dimensional Euclidean space. By $E(n+N, R)$ we denote the Euclidean group of transformations of E^{n+N} over the reals R, i.e. the group whose elements in a fixed co-ordinate system of E^{n+N} can be written in the matrix-form

$$Y = AX + a, \tag{1.1}$$

where $A = \| a_{AB} \|_{1 \leqslant A, B \leqslant n+N}$ denotes an orthogonal matrix and X, Y, a are one-column matrices with (n+N) rows. Transformations (1.1) can be identified with the symbols (A, a) with the following law of composition

$$(C, c) = (B, b) \cdot (A, a) = (BA, Ba + b) \tag{1.2}$$

The Lie algebra g of $E(n+N, R)$ is isomorphic with a subspace spanned over the symbols

$$\left(\frac{\partial}{\partial a_{AB}} \quad \frac{\partial}{\partial a_A} \right)$$

by linear combinations with real coefficients

$$\xi_{AB} \frac{\partial}{\partial a_{AB}} + \xi_A \frac{\partial}{\partial a_A} \tag{1.3}$$

such that

$$\xi_{BA} + \xi_{BA} = 0, \quad \xi_A = a_A, \tag{1.4}$$

the partial derivatives being evaluated at $a_{AB} = \delta_{AB}$, $a_A = 0$. In the sequel employ the summation convention for repeated indices as in (1.3) and we use the following convention concerning indices

$$1 \leqslant i, j, k \leqslant n, \quad n+1 \leqslant r, s, t \leqslant n+N, \quad 1 \leqslant A, B, C \leqslant n+N.$$

By left multiplication the vector (1.3) can be propagated to a left-invariant vector field onto the whole of $E(n+N, R)$. Namly, using (1.2) and taking into account the induced mapping of tangent spaces, we have

$$\xi_{AB} \frac{\partial}{\partial a_{AB}} + \xi_A \frac{\partial}{\partial a_A} \to \xi_{AB} \frac{\partial c_{DE}}{\partial a_{AB}} \frac{\partial}{\partial c_{DE}} + \xi_A \frac{\partial C_D}{\partial a_A} \frac{\partial}{\partial C_D} =$$

$$= b_{AB} \left(\xi_{BC} \frac{\partial}{\partial b_{AC}} + \xi_B \frac{\partial}{\partial b_A} \right).$$

This vector field constitutes the Lie algebra g^*, of E $(n + N, R)$.

Let ω'_A, ω'_{AB} denote the left-invariant linear forms on g* defined by

$$\omega_A' = a_{BA}\, da_B, \quad \omega'_{AB} = a_{CA}\, da_{CB},$$

and

$$da_{AB}\left(\frac{\partial}{\partial a_{CD}}\right) = \delta_{AC}\delta_{BD},\ da_{AB}\left(\frac{\partial}{\partial a_C}\right) = 0,\ da_A\left(\frac{\partial}{\partial a_C}\right) = \delta_{AC},\ da_A\left(\frac{\partial}{\partial a_{BC}}\right) = 0.$$

It follows from (1.4)

$$\omega'_{AB} + \omega'_{BA} = 0$$

The forms ω'_A, ω'_{AB} satisfy the equations of structure of the Euclidean group

(1.5)
$$d\,\omega'_A = \omega'_B \wedge \omega'_{AB}$$
$$d\,\omega'_{AB} = \omega'_{CB} \wedge \omega'_{AC}.$$

2. THE MOVING FRAME. Let

$$x : M^n \to E^{n+N}$$

be an immersion of a closed orientable manifold M^n in E^{n+N}. We consider such elements of E $(n + N, R)$ for which $a^T \in (M^n)$, where a^T denotes the matrix transposed to the matrix a which appears in (1.1) and $e_i = (a_{1i}\, a_{2i}, \ldots a_{n+N,\, i})$ are tangent to the surface $x(M^n)$ at $a^T(p) = x(p)$, $p \in M^n$, and $\det \|a_{AB}\| = 1$, i.e. the frame $x(p) e_1 e_2 \ldots e_{n+N}$ for $e_A = (a_{1A}, a_{2A} \ldots a_{n+N,\, A})$ is oriented coherent with E^{n+N}. Let x^* denote the mapping of differential forms induced by x. We set $\omega_A = x^*\, \omega'_A$, $\omega_{AB} = x^*\, \omega'_{AB}$. Then we have $\omega_r = 0$. This together with (1.5) implies $\omega_{ir} \wedge \omega_r = 0$. Hence

(2.1)
$$\omega_{ir} = A_{rij}\omega_j, \quad A_{rij} = A_{rji}$$

and

$$\Omega_{ij} = \omega_{ir} \wedge \omega_{rj} = A_{rik} A_{rjl} \omega_k \wedge \omega_l = \tag{2.2}$$
$$(A_{rik} A_{rjl} - A_{ril} A_{rjk}) \omega_k \wedge \omega_l = R_{ijkl} \omega_k \wedge \omega_l,$$

where R_{ijkl} is the curvature tensor induced by the immersion.

If Π denotes the plane spanned by two unit orthogonal vectors tangent to the surface $x(M^n)$ at $x(p)$:

$$a = a_i e_i, \quad b = b_i e_i,$$

then the sectional curvature of Π is given by the formula

$$K(p, \Pi) = R_{ijkl}\, a_i\, a_k\, b_j\, b_l.$$

For two-manifolds we have

$$K(p, \Pi) = K(p) = R_{1212},$$

where $K(p)$ denotes the Gauss curvature of $x(M^2)$. Hence we have by (2.2)

$$K(p) = A_{r11} A_{r22} - A_{r12} A_{r12} = \sum_r \det (A_{rij}) . \tag{2.3}$$

3. THE LIPSCHITZ-KILLING CURVATURE. Let ν be an arbitrary unit vector in E^{n+N}. In the following we regard the unit vectors also as points of the unit sphere S^{n+N-1}. Now we define the normal bundle of M^n induced by the immersion x by

$$B_\nu = \{ (p, \nu) / \nu \cdot dx(p) = 0, p \in M^n, \nu \in S^{n+N-1} \}.$$

The fibres of $B_\nu \to M^n$ are $(N-1)$-dimensional unit spheres $S^{N-1}(p)$, and the structural group is the orthogonal group $O(N-1)$. In B_ν we introduce the globally defined differential form

$$d\tau_{n+N-1} = dV_n \wedge d\sigma_{N-1},$$

where $dV_n = \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n$ is the volume element of M^n induced by the immersion x and $d\sigma_{N-1}$ denotes the volume element of the fibre $S^{N-1}(p)$ described by the vector $e_{n+N}(p)$ for a fixed $p \in M^n$. Hence

$$d\sigma_{N-1} = \omega_{n+1,\, n+N} \wedge \omega_{n+2,\, n+N} \wedge \ldots \wedge \omega_{n+N-1,\, n+N}.$$

Let

$$\nu : B_\nu \to S^{n+N-1} \tag{3.1}$$

denote the mapping

$$(p, \nu) \to \nu, \quad (p, \nu) \in B_\nu .$$

The volume element induced by (3.1) in S^{n+N-1} described by e_{n+N} has the form

$$\nu^* d\sigma_{n+N-1} = \omega_{1,\, n+N} \wedge \omega_{2,\, n+N} \wedge \ldots \wedge \omega_{n+N-1,\, n+N}.$$

If we substitute (2.1) for $r = n + N$ into the preceding formula, we have

(3.2) $$\nu^* d\sigma_{n+N-1} = \det(A_{n+N,ij})\, dV_n \wedge d\sigma_{N-1}.$$

We call the function $L(p, e_{n+N}) = \det(A_{n+N,ij})$ the Lipschitz-Killing curvature of B_ν at $(p, e_{n+N}) \in B_\nu$ and the integral

(3.3) $$\int\limits_{S^{N-1}(p)} | L(p, e_{n+N}) |\, d\sigma_{N-1}$$

will be called the Lipschitz-Killing curvature of M^n at $p \in M^n$.

Let e_{n+N} be fixed. The point $p \in M^n$ is called a critical point of the scalar function $-e_{n+N} \cdot x(q)$, $q \in M^n$, if $(p, e_{n+N}) \in B_\nu$, and is called a critical non-degenerated point if the second quadratic form

(3.4) $$-e_{n+N} d^2 x(p) = de_{n+N} dx(p) = A_{n+N,ij}\, \omega_i\, \omega_j$$

of the surface $x(M^n)$ is non-degenerated, i.e. if $\det(A_{n+N,ij}) \neq 0$.
The second differential on the left of (3.4) is taken in the usual (not exterior) sense.

A point $(p, e_{n+N}) \in B_\nu$ for which $\det(A_{n+Nij}) = 0$ is called a critical point of the mapping (3.1). By SARD theorem [2] the set $\nu(Q)$, where

(3.5) $$Q = \{(p, e_{n+N}) \in B_\nu \mid \det(A_{n+N,ij}) = 0\}$$

is of measure zero in S^{n+N-1}. The point (p, e_{n+N}) belongs to Q if and only if $p \in M^n$ is a critical degenerated point of $-e_{n+N} \cdot x(q)$, $q \in M^n$.

Let k denote the index of the function $-e_{n+N} \cdot x(q)$ at a critical non-degenerated point $p \in M^n$, i.e. the maximal dimension of subspaces of the tangent space to $x(M^n)$ for which the quadratic form (3.4) takes negative values. MORSE lemma asserts that in a suitable co-ordinate system introduced in a neighgourhood of p the funcion $f(q) = -e_{n+N} \cdot x(q)$ takes the form

(3.6) $$f(q) = f(p) - t_1^2 - t_2^2 - \ldots - t_k^2 + t_{k+1}^2 + \ldots + t_n^2,$$

where q has the co-ordinates $(t_1, t_2, \ldots, t_n)$. It follows from (3.6) that each non-degenerated critical point is isolated. Hence the number $m_k(M^n, f)$ of critical points of index k of the function f on a closed manifold is finite. Since $\nu(Q)$ is of measure zero in S^{n+N-1}, i follows that in each neighbourhood of an arbitrary vector e^0_{n+N} there exists such a vector e'_{n+N} for which the function $-e'_{n+N} \cdot x(q)$ has only non-degenerated critical points. Moreover, since M^n and S^{n+N-1} are compact and Q is closed, it follows that $\nu(Q)$ is closed, and therefore for each vector e_{n+N} from a small neighbourhood of e_{n+N} the function $-e_{n+N} \cdot x(q)$ will have only non--degenerated critical points. If the function $-e_{n+N} \cdot x(q)$ has index 0 at $p \in M^n$, then $L(p, e_{n+N}) > 0$ and it follows from (3.2) that the induced linear mapping

$$\nu^* : T_{(p,\nu)} \to T_\nu$$

of the tangent space of B_ν onto the tangent space of S^{n+N-1} is orientation-

-preserving for $\nu = e_{n+N}$. If the index of $p \in M^n$ is k, then the orientation defined by ν^* differs by the factor $(-1)^k$ from the positive orientation of S^{n+N-1} defined by the frame $e_{n+N}\, e_1\, e_2 \ldots e_{n+N-1}$ (e_{n+N} denotes a point on S^{n+N-1} which is the origin of $e_1, e_2, \ldots, e_{n+N-1}$). For almost every e_{n+N} the number of all critical points of the function $e_{n+N} \cdot x(q)$ is equal to $m_o + m_1 + \ldots + m_n$, $m_k = m_k(M^n, f)$, $f(q) = -e_{n+N} \cdot x(q)$. Keeping in mind the orientation we have for a point $(p, e_{n+N}) \in B_\nu \setminus Q$

$$\det(A_{n+N,\, ij})\, d\tau_{n+N-1} = (-1)^k\ \nu^* d\sigma_{n+N-1} = (-1)^k d\sigma_{n+N-1},$$

where k denotes the index of $p \in M^n$ with respect to the function $-e_{n+N} \cdot$ $\cdot\, x(q)$, and $d\,\sigma_{n+N-1}$ denotes the positively oriented volume element of the sphere S^{n+N-1}. For a connected neighbourhood $B \subset B \ \setminus Q$ of (p, e_{n+N}) we have therefore

$$\int_B \det(A_{n+N,\, ij})\, d\tau_{n+N-1} = \int_{\nu(B)} (-1)^k\, d\sigma_{n+N-1} \tag{3.7}$$

This equality does not change if we replace B by $B \cup Q$ and $\nu(B)$ by $\nu(B) \cup \nu(Q)$. The set $B \setminus Q$ can be represented as a sum of open disjoint connected sets in each of which equality (3.7) holds for some k $(0 \leqslant k \leqslant n)$. This decomposition leads us to the formula

$$\int_{B_\nu} \det(A_{n+N,\, ij})\, d\tau_{n+N-1} = \int_{S^{n+N-1}} \sum_{k=o}^{n} (-1)^k\, m_k d\sigma_{n+N-1}. \tag{3.8}$$

If b_k denotes the k-th Betti number of M^n, then it follows from the MORSE equality [3]

$$\sum_{k=o}^{n} (-1)^k\, m_k(M^n, f) = \sum_{k=o}^{n} (-1)^k\, b_k(M^n) = \chi(M^n),$$

that we have

$$\int_{B_\nu} \det(A_{n+N,\, ij})\, d\tau_{n+N-1} = v_{n+N-1}\, \chi(M^n), \tag{3.9}$$

where v_{n+N-1} denotes the volume of S^{n+N-1}

If we disregard the orientation, then instead of (3.8) we have

$$\int_{B_\nu} |\det(A_{n+N,\, ij})|\, d\tau_{n+N-1} = \int_{S^{n+N-1}} \sum_{k=o}^{n} m_k\, d\sigma_{n+N-1}. \tag{3.10}$$

Now the MORSE inequalities [3]

$$m_k(M^n, f) \geqslant b_k(M^n) \tag{3.11}$$

imply the following theorem:

THEOREM (S. S. CHERN and R. K. LASHOF [4]). *If the manifold M^n is orientable and closed, then*

$$(3.12) \qquad \int_{B_\nu} |L(p, e_{n+N})| \, d\tau_{n+N-1} \geqslant v_{n+N-1} \sum_{k=o}^{n} b_k.$$

DEFINITION 1. The manifold is said to be immersed in E^{n+N} with minimal total curvature if

$$\int_{B_\nu} |L(p, e_{n+N})| \, d\tau_{n+N-1} = v_{n+N-1} \sum_{k=o}^{n} b_k.$$

Then it follows from (3.10), (3.11) and (3.12) that for almost every $e_{n+N} \in E^{n+N}$ we have

$$(3.13) \qquad m_k(M^n, -e_{n+N} \cdot x(q)) = b_k(M^n).$$

We introduce the following notations:

(3.14) $H(B_\nu) = \{(p, e_{n+N}) \in B_\nu | -e_{n+N} \cdot x(q)$ has index 0 at $p\}$,

$H(M^n)$ denotes the projection $(p, e_{n+N}) \to p$ of $H(B_\nu)$ onto M^n.

The immersion x: $M^n \to E^{n+N}$ with minimal total curvature has the following property.

THEOREM 1. *If* $(p, e_{n+N}) \in H(B_\nu)$, *then the whole surface* $x(M^n)$ *is contained in the halfspace* $\{x \in E^{n+N} | e_{n+N} \cdot x \leqslant e_{n+N} x(p)\}$.

P r o o f. Assume on the contrary that for some $q \in M^n$ the inequality $e_{n+N} \cdot x(q) > e_{n+N} \cdot x(p)$ holds. Since M^n is closed, there exists a point $p_1 \in M^n$ such that the hyperplane $e_{n+N} \cdot x = e_{n+N} \cdot x(p_1)$ is tangent to $x(M^n)$ and for each $q \in M^n$ the inequality $e_{n+N} \cdot x(q) \leqslant e_{n+N} \cdot x(p_1)$ holds. From the definition of p_1 it follows that $e_{n+N} \cdot x(p) \leqslant e_{n+N} \cdot x(p_1)$ and that p_1 is a critical point of the function $-e_{n+N} \cdot x(q)$. If the quadratic form $-e_{n+N} \cdot d^2x$ is non-degenerated, then the function $-e_{n+N} \cdot x(q)$ has index 0 at p_1. If p_1 is a degenrated critical point, then by SARD theorem in an arbitrary neighbourhood of $(p_1. e_{n+N}) \in B_\nu$ there exist points $(p'_1, e'_{n+N}) \in B_\nu$ such that $-e'_{n+N} \cdot x(q)$ has index 0 at p'_1. Since $\nu(Q)$ is closed, there exists a neighbourhood $B \subset B_\nu$ of (p, e_{n+N}) such that for each $(p', e'_{n+N}) \in B$ the function $-e'_{n+N} \cdot x(q)$ has index 0 at p'. Let $d = e_{n+N}$ $(x(p_1) - x(p))$. By the above remarks we can choose a point $(p' e'_{n+N}) \in B_\nu$ such that the following occurs: $-e'_{n+N} x(q)$ has index 0 at p'_1 and for each $q \in M^n$ the inequality $e'_{n+N} \cdot x(q) \leqslant e'_{n+N} \cdot x$ holds. Moreover, there exists a point $p' \in M^n$ such that $(p', e'_{n+N}) \in B$ and $e_{n+N} \cdot (x(p) - x(p')) \leqslant$ $| e_{n+N} \cdot (x(p_1) - x(p'_1)) | \leqslant 1/3\, d$. Thus the function $e'_{n+N} \cdot x(q)$ would have index 0 at two distinet point p', p'_1, ard therefore there would exist a neighbourhood (in S^{n+N-1}) of e'_{n+N} such that for each e_{n+N} belonging to it the function $-e_{n+N} \cdot x(q)$ would have at least two distinct points of index 0. But this contradicts the fact that x is an immersion with minimal total curvature and therefore satisfies (3.13).

If x is an immersion with minimal total curvature then, since M^n is closed and connected, for almost every e_{n+N} we have $m_0(M^n, -e_{n+N} \cdot \cdot x(q)) = 1$. Hence for almost every e_{n+N} there exists exactly one point $p \in M^n$ for which $(p, e_{n+N}) \in H(B_\nu)$ and therefore $L(p, e_{n+N}) > 0$. It follows

$$
\text{(3.15)} \qquad \begin{aligned} v_{n+N-1} &= \int\limits_{H(B_\nu)} L(p, e_{n+N})\, dV_n \wedge d\sigma_{N-1} = \\ &= \int\limits_{H(M^n)} dV_n \int\limits_{h(p)} L(p, e_{n+N})\, d\sigma_{N-1} = \int\limits_{H(M^n)} \overline{L}(p)\, |h(p)|\, dV_n, \end{aligned}
$$

where $h(p) = H(B_\nu) \cap S^{N-1}(p)$, $|h(p)|$ denotes the $(N-1)$-dimensional measure of $h(p)$, $\overline{L}(p)$ denotes the mean value of $L(p, e_{n+N})$ with respect to e_{n+N}.

4. CLOSED SURFACES IN THE EUCLIDEAN FOUR-SPACE. Let

$$x : M^2 \to E^4$$

be an immersion of a closed orientable two-manifold. To avoid additional discussion we assume about x that the following construction is unique: in each fibre $S^1(p)$, $p \in M^2$, we choose such a vector $\bar{e}_4$ that the function $L(p, e_4)$ takes its maximal value for $e_4 = \bar{e}_4$. Then $\bar{e}_3$ is also uniquely determined.

Hence the cross-sections $p \to \bar{e}_3(p)$, $p \to \bar{e}_4(p)$ are defined and B_ν is therefore equivalent to a Cartesian product $M^2 X S^1$. The vector fields $\bar{e}_3(p)$, $\bar{e}_4(p)$ will be called the Frenet frame of M^2 induced by x. From (2.3) and from the definition of the Lipschitz-Killing curvature we have

$$\text{(4.1)} \qquad K(p) = L(p, e_3) + L(p, e_4).$$

It follows from

$$
\begin{aligned} e_3 &= \bar{e}_3 \cos\psi - \bar{e}_4 \sin\psi \\ e_4 &= \bar{e}_3 \sin\psi + \bar{e}_4 \sin\psi, \end{aligned} \qquad 0 \leqslant \psi < 2\pi
$$

$$\text{(4.2)} \qquad d\tau_3 = \omega_1 \wedge \omega_2 \wedge \omega_{34} = \omega_1 \wedge \omega_2 \wedge (\omega_{34} + d\psi) = \omega_1 \wedge \omega_2 \wedge d\psi.$$

that $\omega_{34} = de_4 \cdot e_3 = \overline{\omega}_{34} + d\psi$. Therefore we have

Using (3.15) we get

$$\text{(4.3)} \qquad \int\limits_{H(B_\nu)} L(p, e_4)\, dV_2 \wedge d\psi = 2\pi^2,$$

$$
\text{(4.4)} \qquad \begin{aligned} &\int\limits_{H(B_\nu)} (L(p, e_3) + L(p, e_4))\, dV_2 \wedge \omega_{34} = \int\limits_{H(B_\nu)} K(p)\, dV_2 \wedge d\psi = \\ &\int\limits_{H(M^2)} dV_2 \int\limits_{h(p)} K(p) d\psi = \int\limits_{H(M^2)} K(p)\, |h(p)|\, dV_2 \end{aligned}
$$

if x is an immersion with minimal total curvature. The function $|h(p)|$ is positive for $p \in H(M^2)$. This follows from the fact that $H(B_\nu)$ is open and therefore for each $(p, e_4) \in H(B_\nu)$ there exists a neighbourhood $B \subset H(B_\nu)$ of this point and the set $B \cap S^1(p)$ is open in $S^1(p)$ and is not empty.

In the following x is an immersion with minimal total curvature and g denotes the genus of M^2. Let e_4 be an arbitrary unit vector, then for almost every e_4 the function $e_4 \cdot x(p)$ has exactly $(2+2g)$ critical non--degenerated points

$$p_1, p_2, \ldots, p_{2+2g}, \tag{4.5}$$

where M^2 has genus g. It follows from the definition of a critical point that e_4 is orthogonal to $x(M^2)$ at $x(p_\alpha)$ $(1 \leqslant \alpha \leqslant 2+2g)$. Besides e_4 there exists for every α a unit vector $e_3(p_\alpha)$ which is orthogonal to $x(M^2)$ at $x(p_\alpha)$ and to e_4 and such that the frame $x(p_\alpha)\, e_1 e_2 e_3(p_\alpha)\, e_4$ determines an orientation coherent with that of E^4. Hence p_α is also a critical point for the function $e_3(p_\alpha) \cdot x(q)$. Since p_α is a critical non-degenerated point of $e_4 \cdot x(q)$, there exists a connected neighbourhood $B_\alpha \subset B$ of (e_4, p'_α) such that if $(e'_4, p'_\alpha) \in B_\alpha$ then p'_α is a non-degenerated critical point of $e'_4 \cdot x(q)$. Moreover we can assume that $B_\alpha \cap B_\beta = \emptyset$ for $\alpha \neq \beta$ $(1 \leqslant \leqslant \alpha, \beta \leqslant 2+2g)$. Since the mapping ν (see (3.1)) is locally a diffeomorphism, we can suppose that $S_\alpha = \nu(B_\alpha)$ is open in S^3. One can easily verify that $e_4 \in S = S_1 \cap S_2 \cap \ldots \cap S_{2+2g}$ and the function $e'_4 \cdot x(q)$ has only non-degenerated critical points for every $e'_4 \in S$. We define $B_\alpha = \nu^{-1}(S)$. Since $e'_3(p_\alpha)$ is uniquely determined by e_4, p_α and the orientation of E^4, we define the neighbourhood B'_α of $(p_\alpha, e_3(p_\alpha))$ to be the set of all pairs $(p'_\alpha, e'_3\, p'_\alpha)$ such that $(p'_\alpha, e'_4) \in B_\alpha$ and $e'_3(p'_\alpha)$ is the complementary vector of e'_4. Since the mapping $(p, e_4) \to (p, e_3)$, (p, e_3), $(p, e_4) \in B$, is an automorphism, the set B'_α is open and connected. If p_α is a non-degenerated critical point of $e_3(p_\alpha) \cdot x(q)$, then let $B''_\alpha \subset B'_\alpha$ denote a neighbourhood of $((p_\alpha, e_3(p_\alpha))$ such that for every $(p''_\alpha, e''_3)\, p''_\alpha)) \in B''_\alpha$ the point p''_α is a non-degenerated critical point of $e''_3(p_\alpha) \cdot x(q)$. Now, if p is a degenerated critical point of $e_3(p_\alpha) \cdot x(q)$, then in virtue of Sard theorem there exists a vector $e'_4 \in S$ such that for each α $(1 \leqslant \alpha \leqslant 2+2g)$ we have $(p'_\alpha, e'_4) \in B_\alpha$ and for γ $(1 \leqslant \gamma \leqslant 2+2g)$ such that B''_γ is defined, i. e. p'_α is a non-degenerated critical point of $e'_3(p'_\alpha) . x(q)$, we have $(p'_\alpha, e'_3(p'_\alpha)) \in B''_\gamma$, and p'_β is a non-degenerated critical point of $e'_3(p'_\beta) \cdot x(q)$. Thus we get after a finite number of steps: If x is an immersion with minimal total curvature of M^2 in E^4 and $e_4 \in S^3$, then in every neighbourhood $S \subset S^3$ of e_4 there exists a vector $e'_4 \in S$ such that for each α $(1 \leqslant \alpha \leqslant 2+2g)$ p_α is a critical non-degenerated point of $e'_4 \cdot x(q)$ as well as of $e'_3(p_\alpha) \cdot x(q)$. Moreover, since there are a finite number of points p_α wheach are critical points of $e_4 \cdot x(q)$, and $e_3(p_\alpha) \cdot \cdot x(q)$ we obtain the following.

LEMMA. *The set of points $e_4 \in S^3$ for which not all p_α are critical non-degenerated points of $e_4 \cdot x(q)$, $e_3(p_\alpha) \cdot x(q)$ is of measure zero in S^3.*

By the above lemma we can suppose that each p_α of (4.5) is a critical non-degenerated point of both $e_4 \cdot x(q)$ and $e_3(p_\alpha) \cdot x(q)$. The points (4.5) can be split into three classes: μ_0, μ_1, μ_2 in the following manner: $p_\alpha \in \mu_k$ $(k = 0, 1, 2)$ if p_α is of index k of $e_3(p_\alpha) \cdot x(q)$. By m'_k we denote the cardinal number of μ_k, i.e. $\mu_k = m'_k$. We define

$$\chi'(M^2, e_4) = m'_0 - m'_1 + m'_2.$$

DEFINITION 2. The immersion $x : M^2 \to E^4$ with minimal total curvature is called rigid, if for almost every $e_4 \in S^3$ we have

(4.6) $$m'_0 = m_0 = 1.$$

If x is rigid, then for almost every e_4 we have $\chi'(M^2, e_4) = \chi(M^2)$. Indeed, it follows from (4.6) that $m'_2 = 1$ and therefore $m'^1 = 2g$.

THEOREM 2. *If $x : M^2 \to E^4$ is rigid, then there exists such a point $p \in M^2$ for which the Gauss curvature $K(p)$ of the metric induced by the immersion is non-negative*

P r o o f. By (4.4) it suffices to prove the inequality

(4.7) $$\int_{H(B_\nu)} K(p)\, dV_2 \wedge d\psi \geqslant 0.$$

It follows from (4.1)

$$K(p_\alpha) = L(p_\alpha, e_3(p_\alpha)) + L(p, e_4).$$

Let $Y(S^3)$ denote the space of all $(2 + 2g)$-point sequences of S^3. The distance between two sequences is defined as the Hausdorff dinstance between their coresponding point sets.

For almost every $e_4 \in S^3$ we define the function

(4.8) $$F(e_4) = (e_3(p_1), e_3(p_2), \ldots, e_3(p_{2+2g})),$$

where p_α is a critical non-degenerated point of $e_4 \cdot x(q)$ and $e_3(p_\alpha) \cdot x(q)$. We are going to show that $F(S^3)$ can be identified with a $(2 + 2g)$-covering of S^3, i.e. every point of S^3 is covered exactly $(2+2g)$ times by the values of F, except for a set of measure zero in S^3.

Indeed, let e_3 be such that $e_3 \cdot x(q)$ has only non-degenerated critical points p_α $(1 \leqslant \alpha \leqslant 2+2g)$ and p_α is a critical non-degenrated point of $e_4(p_\alpha) \cdot x(q)$, where $e_4(p_\alpha)$ is orthogonal to the surface $x(M^2)$ at $x(p_\alpha)$ and to e_3 and the frame $x(p_\alpha)\, e_1\, e_2\, e_3\, e_4(p_\alpha)$ determines the positive orientation of E^4. By the lemma we can assume that F is defined for $e_4(p_\alpha)$ up to a small change of e_3. From the construction of $e_4(p_\alpha)$ it follows that in the image-sequence $F(e_4(p_\alpha))$, which is of form (4.8), the vector e_3 appears. Hence the point $e_3 \in S^3$ is covered exactly $(2+2g)$ times (except for a set of measure zero) when e_4 describes S^3. Since x is an immersion with minimal total curvature, S^3 is covered twice (up to a set of measure zero) by points $(p, e_4) \in B_\nu$ for which the function $-e_4 \cdot x(q)$ has index 0 or 2. The mapping ν reduced to $H(B_\nu)$, i. e. to the set of points of index 2, is orientation preserving, since at such points the Lipschitz--Kiling curvature is positive (see section 3). S^3 is covered $2g$ times by points for which the function mentioned has index 1. Hence the Lipschitz--Killing curvature has negative values at such points and then ν is orientation reversing. Let $e_3(p_1)$ denote this vector of the image-sequence (4.8) for which $-e_4 \cdot x(q)$ has index 0 at p_1. From (3.14) we have $(p_1, e_4) \in H(B_\nu)$. We define

$$F(e_4) = e_3(p_1)$$

for almost every $e_4 \in S^3$.

Now we prove that no part of positive measure of S^3 is covered twice by $e_3(p_1)$ when e_4 ranges over the possible values of S^3. Assume the contrary and suppose that a fixed $e_3(p_1)$ belongs to such a part. Since the part considered is of positive measure, we can choose $e_3(p_1)$ in such a manner that $F(e_3(p_1))$ is defined. Suppose

$$F(e_3(p_1)) = (e_4(q_1), e_4(q_2), \ldots, e_4(q_{2+2g})). \tag{4.9}$$

Since $e_3(p_1)$ is covered at least twice and $F(e_4(q_\alpha))$ are the only image--sequences in which $e_3(p_1)$ appears, we have for at least two distinct indices α, β $(1 \leqslant \alpha, \beta \leqslant 2+2g)$

$$F(e_4(q_\alpha)) = F(e_4(q_\beta)) = e_3(p_1).$$

From the definition of the function F it follows that q_α, q_β are distinct non-degenerated critical points of index 0 of the functions $-e_4(q_\alpha) \cdot x(q)$, $-e_4(q_\beta) \cdot x(q)$, respectively. Hence for the image-sequence (4.9) we would have $m'_0 \geqslant 2$. But this contradicts the fact that x is a rigid immersion. Thus we have proved that

$$\int\limits_{H(B_\nu)} L(p_1, e_3(p_1))\, dV_2 \wedge d\psi \geqslant -2\pi^2.$$

From (4.3), from the above inequality and from the definition of the Gauss curvature we obtain (4.7).

REFERENCES

[1] **S. S. Chern and R. K. Lasof:** ***On the total curvature of immersed manifolds.*** **Amer. Journ. of Math. 79 (1957), 306—318.**

[2] A. Sard: *The measure of the critical points of differentiable maps.* Bull. Amer. Math. Soc. 48 (1942), 883—890.

[3] **H. Seifert und W. Threlfall:** ***Variationsrechnung im grossen,*** **New York (1948).**

[4] **S. S. Chern and R. K. Lashof:** ***On the total curvature of immersed manifolds II,*** **Michigan Math. Journ. 5 (1958), 5—12.**

MAREK ROCHOWSKI

ZANURZENIA ROZMAITOŚCI DWUWYMIAROWYCH W PRZESTRZEŃ EUKLIDESOWĄ CZTEROWYMIAROWĄ

Streszczenie

W pracy podane są warunki dostateczne na to, żeby zanurzenie rozmaitości dwuwymiarowej, zamkniętej i orientowalnej w przestrzeń euklidesową czterowymiarową indukowało na niej metrykę o krzywiźnie Gaussa nie wszędzie ujemnej. Wynika stąd, że zanurzenia takie nie mogą być izometriami dla rozmaitości rodzaju $\geqslant 2$ rozważanych jako przestrzenie lokalnie izometryczne z płaszczyzną nieeuklidesową.

Oddano do Redakcji 20. 4. 70