PRACE NAUKOWE UNIWERSYTETU ŚLĄSKIEGO W KATOWICACH NR 12
PRACE MATEMATYCZNE II, 1972

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Differential concomitants of the metrical tensor in Riemannian spaces C^n and S^n

INTRODUCTION. Let V^n be an n-dimensional Riemannian space. The metrical tensor on V^n will be denoted by g_{ij}, Det $g_{ij} \neq 0$.

A space V^n, n > 3, is *conformally Euclidean* if

$$R_{ijkl} = \frac{1}{n-2} (R_{il} g_{jk} + R_{jk} g_{il} - R_{ik} g_{jl} - R_{jl} g_{ik}) + \tag{1}$$
$$+ \frac{R}{(n-1)(n-2)} (g_{jl} g_{ik} - g_{jk} g_{il}),$$

where R_{ijkl} is the curvature tensor, $R_{jk} = g^{rs} R_{rjks}$ in the Ricci tensor and R is the scalar curvature of the space V^n (cf. [6] p. 521). The conformally Euclidean space V^n will be denoted by C^n.

It is known (cf. [2] p. 73, also [7] p. 138) that every scalar differential concomitant of the second order of the tensor g_{ij} is an algebraic concomitant of the curvature tensor and of the tensor g_{ij}. For $n = 4$ the problem of the determination of such concomitants was considered in papers [4] and [5].

In the present paper we shall consider the spaces C^n for arbitrary $n > 3$ (for V^2 and V^3 all scalar differential concomitants were found in papers [2] and [3]).

In § 1 we shall determine the general form of differential concomitants of the second order of g_{ij} which are scalars.

In § 2 we shall deal with this problem for spaces of constant curvature.

§ 1. Let us consider a space C^n. The problem of finding scalar differential concomitants of the second order of the tensor g_{ij} reduces to solving the equation

$$\omega (g_{ij}, R_{ijkl}) = \omega (g_{i'j'}, R_{j'i'k'l'}) \tag{1.1}$$
$$i, j, k, l, i', j, k', l' = 1,2, \ldots, n .$$

Since in C^n the tensor R_{ijkl} satisfies condition (1), our problem leads to the determination of the scalar concomitants of the pair of tensors (g_{ij}, R_{ij}).

We consider the tensor bundle

$$g_{ij}\,\lambda + R_{ij} \tag{1.2}$$

and the matrix bundle

$$G\lambda + H, \tag{1.3}$$

where $G = \| g_{ij} \|$, $H = \| R_{ij} \|$. We denote by $R_1, R_2, \ldots, R_n$ the coefficients of the characteristic polynomial of $\| R^i{}_j \| = \| g^{is} R_{sj} \|$ and we denote by σ the partial signatures (Teilsignaturen) of the canonical form of bundle (1.3) (cf. [8] p. 18).

We shall prove the following theorem

THEOREM 1. *Every scalar differential concomitant of the second order of the metrical tensor on C^n is an arbitrary function of the partial signatures σ, of the scalars $R_1, R_2, \ldots, R_n$, and of the Weierstrass 's characteristic of the matrix $\| R^i{}_j \|$.*

P r o o f. Since the bundle $G\lambda + H$ is regular, i. e. Det $(G\lambda + H) \neq 0$, every scalar concomitant of the tensor bundle $g_{ij}\lambda + R_{ij}$ is a function of the elementary divisors and of the partial signatures of bundle (1.3) (cf. [8] p. 22). The bundles $G\lambda + H$ and $E\lambda + G^{-1}H$ are stricly equivalent (cf. [1] p. 148). Hence these bundles have identical elementary divisors (cf. [1], p. 332). Te elementary divisors are determined by $R_1, R_2, \ldots, R_n$ and by the Weierstrass's characteristic $[e_1, e_2, \ldots, e_n]$ of $\| R^i{}_j \|$. Therefore an arbitary scalar differential concomitant ot the second order of the tensor g_{ij} has the form

$$\varrho\,(\sigma, R_1, R_2, \ldots, R_n, [e_1, e_2, \ldots, e_n]). \tag{1.4}$$

This completes the proof.

A space V^n is called ordinary if the tensor g_{ij} is positive definite. From theorem 1 we obtain the following conclusion:

COROLLARY 1. *If C^n is ordinary, then every scalar differential concomitant of the second order of the metrical tensor is a function of the scalars $R_1, R_2, \ldots, R_n$.*

§ 2. A space V^n, $n > 2$, is called S^n or a *space of constant curvature*, if

$$R_{ijkl} = K\, g_{ijkl}, \tag{2.1}$$

where $g_{ijkl} \overset{\mathrm{df}}{=} g_{[i[k}\, g_{j]l]}$ (cf. [6] p. 501).

We shall prove the following theorem.

THEOREM 2. *If a V^n is S^n, then every scalar differential concomitant of the second order the metrical tensor is an arbitrary function of the curvature K and of the signature of the metrical tensor.*

P r o o f. Every scalar differential concomitant of the second order of the tensor g_{ij} is an algebraic concomitant of the tensors g_{ij} and R_{ijkl}.

Thus it must satisfy equation (1.1). Since in an S^n tensor R_{ijkl} satisfies condition (2.1), our problem leads to the solution of the equation

$$\omega (g_{ij}, K) = \omega (g_{i'j'}, K). \tag{2.2}$$

It is known (cf. [1] p. 341) that we may always find a non-singular matrix $\| B^{i}_{i'} \|$ such that

$$\| g_{i'j'} \| = \| B^{i}_{i'} B^{j}_{j'} g_{ij} \| = \begin{Vmatrix} \varepsilon_1, & 0, & \dots & 0 \\ 0, & \varepsilon_2, & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots & \varepsilon_n \end{Vmatrix} \tag{2.3}$$

where $(\varepsilon_i)^2 = 1$. Inserting relation (2.3) into equation (2.2) we obtain

$$\omega (g_{ij}, K) = \omega \left(\begin{Vmatrix} \varepsilon_1, & 0, & \dots & 0 \\ 0, & \varepsilon_2, & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots & \varepsilon_n \end{Vmatrix}, K \right). \tag{2.4}$$

Thus $\omega (g_{ij}, K) = \varrho (s, K)$, where s is the signature of the tensor g_{ij}. This completes the proof.

THEOREM 3. *If V^n is S^n, then there do not exist differential concomitants of the second order of the metrical tensor which are G-densities of a weight* p, $\mathrm{p} \neq 0$.

THEOREM 4. *If V^n is S^n, then every differential concomitant of the second order of the metrical tensor which is a W-density of a weight* p *has the form*

$$| g | ^{-\frac{\mathrm{p}}{2}} \varrho (s, K)$$

where $g = \mathrm{Det}\ \| g_{ij} \|$ and $\varrho (s, K)$ *is an arbitrary scalar differential concomitant of the second order of the tensor* g_{ij}.

The proofs of theorems 3 and 4 are quite similar to the proofs of theorems 4 and 5 in paper [2].

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KOMITANTY RÓŻNICZKOWE TENSORA METRYCZNEGO W PRZESTRZENIACH RIEMANNA C^n i S^n

Streszczenie

W pracy zajmuję się problemem wyznaczania komitant różniczkowych drugiego rzędu tensora metrycznego w przestrzeniach konforemnie płaskich C^n i w przestrzeniach o stałej krzywiźnie S^n.

W § 1 została znaleziona ogólna postać komitant skalarnych. Wykazano mianowicie, że każda taka komitanta ma postać

$$\varrho\,(\sigma,\ R_1,\ R_2, \ldots,\ R_n,\ [e_1 \ldots e_n])$$

gdzie σ oznacza sygnatury częściowe pęku macierzy $G\lambda + K$, $R_1, R_2, \ldots, R_n$ są współczynnikami wielomianu charakterystycznego tensora $R^i{}_k = g^{is} R_{sk}$, a $[e_1 \ldots e_n]$ jest charakterystyką Weierstrassa macierzy $\| R^i{}_k \|$.

W § 2 wykazano, że skalarne komitanty różniczkowe drugiego rzędu tensora metrycznego w przestrzeniach o stałej krzywiźnie są postaci

$$\varrho\,(s,\ K),$$

gdzie s jest sygnaturą tensora metrycznego a K oznacza krzywiznę skalarną tej przestrzeni. Znaleziona została ogólna postać komitant różniczkowych drugiego rzędu będących W-gęstościami i wykazano, że nie istnieją takie komitanty, które są G-gęstościami.

Oddano do Redakcji 17. 12. 1969