



You have downloaded a document from
RE-BUŚ
repository of the University of Silesia in Katowice

Title: Continuous solutions to two iterative functional equations

Author: Karol Baron

Citation style: Baron Karol. (2021). Continuous solutions to two iterative functional equations. "Aequationes mathematicae" (2021), doi 10.1007/s00010-021-00794-x



Uznanie autorstwa - Licencja ta pozwala na kopiowanie, zmienianie, rozprowadzanie, przedstawianie i wykonywanie utworu jedynie pod warunkiem oznaczenia autorstwa.



Continuous solutions to two iterative functional equations

KAROL BARON 

Dedicated to Professor Ludwig Reich on his 80th birthday.

Abstract. Based on iteration of random-valued functions we study the problem of solvability in the class of continuous and Hölder continuous functions φ of the equations

$$\begin{aligned}\varphi(x) &= F(x) - \int_{\Omega} \varphi(f(x, \omega)) P(d\omega), \\ \varphi(x) &= F(x) + \int_{\Omega} \varphi(f(x, \omega)) P(d\omega),\end{aligned}$$

where P is a probability measure on a σ -algebra of subsets of Ω .

Mathematics Subject Classification. Primary 39B12; Secondary 60B12, 26A16, 54H05.

Keywords. Iterative functional equations, Continuous and Hölder continuous solutions, Random-valued functions, Iterates, Convergence in law, Dense sets, Sets of first category, Haar zero sets.

1. Introduction

Fix a probability space (Ω, \mathcal{A}, P) , a complete and separable metric space (X, ρ) with the σ -algebra \mathcal{B} of all its Borel subsets, and a $\mathcal{B} \otimes \mathcal{A}$ -measurable function $f : X \times \Omega \rightarrow X$.

We continue the research of continuous solutions $\varphi : X \rightarrow \mathbb{R}$ of the equations

$$\varphi(x) = F(x) - \int_{\Omega} \varphi(f(x, \omega)) P(d\omega), \quad (1)$$

$$\varphi(x) = F(x) + \int_{\Omega} \varphi(f(x, \omega)) P(d\omega). \quad (2)$$

We refer mainly to [2, 5]. Like in these papers we focus on the iteration of random-valued functions:

$$f^0(x, \omega_1, \omega_2, \dots) = x, \quad f^n(x, \omega_1, \omega_2, \dots) = f(f^{n-1}(x, \omega_1, \omega_2, \dots), \omega_n)$$

for $n \in \mathbb{N}$, $x \in X$ and $(\omega_1, \omega_2, \dots)$ from Ω^∞ defined as $\Omega^{\mathbb{N}}$. Note that for $n \in \mathbb{N}$ the n th iterate f^n mapping $X \times \Omega^\infty$ into X is $\mathcal{B} \otimes \mathcal{A}_n$ -measurable, where \mathcal{A}_n denotes the σ -algebra of all sets of the form

$$\{(\omega_1, \omega_2, \dots) \in \Omega^\infty : (\omega_1, \dots, \omega_n) \in A\}$$

with A from the product σ -algebra \mathcal{A}^n . (See [13, section 1.4], [10].)

Let $\pi_n^f(x, \cdot)$ denote the distribution of $f^n(x, \cdot)$, i.e.,

$$\pi_n^f(x, B) = P^\infty(f^n(x, \cdot) \in B) \quad \text{for } n \in \mathbb{N} \cup \{0\}, \quad x \in X \text{ and } B \in \mathcal{B}.$$

If

$$\int_{\Omega} \rho(f(x, \omega), f(z, \omega)) P(d\omega) \leq \lambda \rho(x, z) \quad \text{for } x, z \in X \quad (3)$$

with a $\lambda \in (0, 1)$, and

$$\int_{\Omega} \rho(f(x, \omega), x) P(d\omega) < \infty \quad \text{for } x \in X, \quad (4)$$

then (see [2, Theorem 3.1]) there exists a probability Borel measure π^f on X such that for every $x \in X$ the sequence $(\pi_n^f(x, \cdot))_{n \in \mathbb{N}}$ converges weakly to π^f , and (see [11, Corollary 5.6 and Lemma 3.1], also [3, Lemma 2.2]) for every non-expansive $u : X \rightarrow \mathbb{R}$ the inequality

$$\left| \int_X u(z) \pi_n^f(x, dz) - \int_X u(z) \pi^f(dz) \right| \leq \frac{\lambda^n}{1 - \lambda} \int_{\Omega} \rho(f(x, \omega), x) P(d\omega) \quad (5)$$

holds for $x \in X$ and $n \in \mathbb{N}$.

This limit distribution π^f plays an important role in solving (1) and (2), see [5, Theorem 3.1], [2, Corollary 4.1], [3, Theorem 2.1]. In particular:

(I) If $F : X \rightarrow \mathbb{R}$ is continuous and bounded, then any continuous and bounded solution $\varphi : X \rightarrow \mathbb{R}$ of (1) has the form

$$\begin{aligned} \varphi(x) = & F(x) - \frac{1}{2} \int_X F(z) \pi^f(dz) \\ & + \sum_{n=1}^{\infty} (-1)^n \left(\int_X F(z) \pi_n^f(x, dz) - \int_X F(z) \pi^f(dz) \right) \quad \text{for } x \in X; \end{aligned} \quad (6)$$

if additionally F is Lipschitz, then (6) defines a Lipschitz solution $\varphi : X \rightarrow \mathbb{R}$ of (1).

(II) If $F : X \rightarrow \mathbb{R}$ is continuous and bounded and (2) has a continuous and bounded solution $\varphi : X \rightarrow \mathbb{R}$, then

$$\int_X F(x) \pi^f(dx) = 0, \quad (7)$$

and any such solution has the form

$$\varphi(x) = c + F(x) + \sum_{n=1}^{\infty} \int_X F(z) \pi_n^f(x, dz) \quad \text{for } x \in X$$

with a real constant c .

(III) If $F : X \rightarrow \mathbb{R}$ is Lipschitz, then it is integrable for π^f and (2) has a Lipschitz solution $\varphi : X \rightarrow \mathbb{R}$ if and only if (7) holds.

The limit distribution π^f and facts cited above will be used in the main part of the paper. A characterization of this limit for some special random-valued functions in Hilbert spaces have been given by [3, Theorem 3.1] and, in Banach spaces, by [4, Theorem 2.1].

Actually we do not have a sufficiently satisfactory theorem to guarantee the existence of continuous solutions to the equations considered. An explanation of this situation is given in the paper [9] by Witold Jarczyk (see also [13, Note 3.8.4]). Namely, in the case where Ω is a singleton and X is a compact real interval, for the appropriate f the set of continuous $F : X \rightarrow \mathbb{R}$ such that the equation has a continuous solution is small in the sense of Baire category. It is also small from the measure point of view (see [1]). We will go also in this direction but, above all, we are looking for conditions under which Eqs. (1) and (2) have continuous and Hölder continuous solutions $\varphi : X \rightarrow \mathbb{R}$. In the case where Ω is a singleton, see [12, Chapter II, §7].

2. Results

We will consider Eqs. (1) and (2) assuming the following hypothesis (H).

(H) (Ω, \mathcal{A}, P) is a probability space, (X, ρ) is a complete and separable metric space, $f : X \times \Omega \rightarrow X$ is $\mathcal{B} \otimes \mathcal{A}$ -measurable, (3) holds with a $\lambda \in (0, 1)$ and (4) is satisfied.

We regard λ as fixed in $(0, 1)$, and for any metric space X we define $\mathcal{F}(X)$ as the set of all continuous functions $F : X \rightarrow \mathbb{R}$ such that there are a sequence $(F_n)_{n \in \mathbb{N}}$ of real functions on X and constants $\vartheta \in (0, 1)$, $L \in (0, \frac{1}{\lambda})$ and $\alpha, \beta \in (0, \infty)$ such that

$$|F(x) - F_n(x)| \leq \alpha \vartheta^n \quad \text{for } x \in X, n \in \mathbb{N},$$

and

$$|F_n(x) - F_n(z)| \leq \beta L^n \rho(x, z) \quad \text{for } x, z \in X, n \in \mathbb{N}.$$

Clearly any real Lipschitz function defined on X belongs to $\mathcal{F}(X)$.

Theorem 2.1. *Assume (H). If $F \in \mathcal{F}(X)$, then formula (6) defines a continuous solution $\varphi : X \rightarrow \mathbb{R}$ of (1), and if additionally (7) holds, then the formula*

$$\varphi_0(x) = F(x) + \sum_{n=1}^{\infty} \int_X F(z) \pi_n^f(x, dz) \quad \text{for } x \in X \quad (8)$$

defines a continuous solution $\varphi_0 : X \rightarrow \mathbb{R}$ of (2).

The proof will be based on three lemmas. In each of them we assume (H).

Lemma 2.2. *If $F \in \mathcal{F}(X)$, then the integrals*

$$\int_{\Omega} |F(f(x, \omega))| P(d\omega) \quad \text{for } x \in X, \quad \int_X |F(z)| \pi^f(dz)$$

are finite, and the function

$$x \mapsto \int_{\Omega} F(f(x, \omega)) P(d\omega), \quad x \in X, \quad (9)$$

is continuous.

Proof. Corresponding to F choose a sequence $(F_n)_{n \in \mathbb{N}}$ of real functions on X and constants $\vartheta \in (0, 1)$, $L \in (0, \frac{1}{\lambda})$ and $\alpha, \beta \in (0, \infty)$ as in the definition of $\mathcal{F}(X)$. Then

$$\int_{\Omega} |F(f(x, \omega))| P(d\omega) \leq \alpha \vartheta + \beta L \int_{\Omega} \rho(f(x, \omega), x) P(d\omega) + |F_1(x)|$$

for $x \in X$, and

$$\int_X |F(z)| \pi^f(dz) \leq \alpha \vartheta + \int_X |F_1(z)| \pi^f(dz),$$

see also (III). Moreover, for every $n \in \mathbb{N}$ the function

$$x \mapsto \int_{\Omega} F_n(f(x, \omega)) P(d\omega), \quad x \in X,$$

is Lipschitz:

$$\left| \int_{\Omega} F_n(f(x, \omega)) P(d\omega) - \int_{\Omega} F_n(f(z, \omega)) P(d\omega) \right| \leq \beta L^n \lambda \rho(x, z)$$

for $x, z \in X$, and therefore function (9), as their uniform limit, is continuous. \square

Lemma 2.3. *If $F \in \mathcal{F}(X)$, then*

$$\int_X |F(z)| \pi_n^f(x, dz) < \infty \quad \text{for } x \in X \text{ and } n \in \mathbb{N},$$

and for every $n \in \mathbb{N}$ the function

$$x \mapsto \int_X F(z) \pi_n^f(x, dz), \quad x \in X,$$

is continuous.

Proof. By induction, (3) and (4),

$$\int_{\Omega^\infty} \rho(f^n(x, \omega), f^n(z, \omega)) P^\infty(d\omega) \leq \lambda^n \rho(x, z) \quad \text{for } x, z \in X \text{ and } n \in \mathbb{N} \quad (10)$$

and

$$\int_{\Omega^\infty} \rho(f^n(x, \omega), x) P^\infty(d\omega) < \infty \quad \text{for } x \in X \text{ and } n \in \mathbb{N}.$$

Since

$$\int_X F(z) \pi_n^f(x, dz) = \int_{\Omega^\infty} F(f^n(x, \omega)) P^\infty(d\omega) \quad \text{for } x \in X \text{ and } n \in \mathbb{N}, \quad (11)$$

an application of Lemma 2.2 with f replaced by f^n , $n \in \mathbb{N}$, finishes the proof. \square

Lemma 2.4. *If $F \in \mathcal{F}(X)$, then there are constants $\theta \in (0, 1)$ and $M \in (0, \infty)$ such that*

$$\begin{aligned} & \left| \int_X F(z) \pi_n^f(x, dz) - \int_X F(z) \pi^f(dz) \right| \\ & \leq M\theta^n (1 + \rho(x, x_0) + \int_{\Omega} \rho(f(x_0, \omega), x_0) P(d\omega)) \end{aligned}$$

for $x, x_0 \in X$ and $n \in \mathbb{N}$.

Proof. Corresponding to F choose a sequence $(F_n)_{n \in \mathbb{N}}$ of real functions on X and constants $\vartheta \in (0, 1)$, $L \in (0, \frac{1}{\lambda})$ and $\alpha, \beta \in (0, \infty)$ as in the definition of $\mathcal{F}(X)$, and put

$$\theta = \max\{\vartheta, \lambda L\}, \quad M = 2 \max \left\{ \alpha, \frac{\beta}{1 - \lambda} \right\}.$$

Then $\theta \in (0, 1)$, and by Lemmas 2.3 and 2.2, (5) with $u = \frac{F_n}{\beta L^n}$ and (3) for every $x, x_0 \in X$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
& \left| \int_X F(z) \pi_n^f(x, dz) - \int_X F(z) \pi^f(dz) \right| \\
& \leq \left| \int_X F(z) \pi_n^f(x, dz) - \int_X F_n(z) \pi_n^f(x, dz) \right| \\
& \quad + \left| \int_X F_n(z) \pi_n^f(x, dz) - \int_X F_n(z) \pi^f(dz) \right| \\
& \quad + \left| \int_X F_n(z) \pi^f(dz) - \int_X F(z) \pi^f(dz) \right| \\
& \leq \int_X |F(z) - F_n(z)| \pi_n^f(x, dz) + \beta L^n \frac{\lambda^n}{1-\lambda} \int_\Omega \rho(f(x, \omega), x) P(d\omega) \\
& \quad + \int_X |F_n(z) - F(z)| \pi^f(dz) \leq 2\alpha \theta^n + \beta L^n \frac{\lambda^n}{1-\lambda} \int_\Omega \rho(f(x, \omega), x) P(d\omega) \\
& \leq 2\alpha \theta^n + \frac{\beta}{1-\lambda} \theta^n (\lambda \rho(x, x_0) + \int_\Omega \rho(f(x_0, \omega), x_0) P(d\omega) + \rho(x, x_0)) \\
& \leq M \theta^n (1 + \rho(x, x_0) + \int_\Omega \rho(f(x_0, \omega), x_0) P(d\omega)).
\end{aligned}$$

□

Proof of Theorem 2.1. It follows from Lemmas 2.2–2.4 that formula (6) defines a continuous function $\varphi : X \rightarrow \mathbb{R}$ and arguing like in the proof of Theorem 3.1(ii) of [5] (see also the calculations below) we show that it solves (1).

Assume now that also (7) holds. Then it follows from Lemmas 2.3 and 2.4 that formula (8) defines a continuous function $\varphi_0 : X \rightarrow \mathbb{R}$. Applying (11), Lemma 2.4, the Lebesgue dominated convergence theorem and the Fubini theorem we observe that for every $x \in X$ the function $\varphi_0 \circ f(x, \cdot)$ is integrable for P and

$$\begin{aligned}
& \int_\Omega \varphi_0(f(x, \omega)) P(d\omega) = \int_\Omega F(f(x, \omega)) P(d\omega) \\
& \quad + \int_\Omega \sum_{n=1}^{\infty} \left(\int_X F(z) \pi_n^f(f(x, \omega), dz) \right) P(d\omega) = \int_X F(z) \pi_1^f(x, dz) \\
& \quad + \sum_{n=1}^{\infty} \int_\Omega \left(\int_{\Omega^\infty} F(f^n(f(x, \omega_1), \omega_2, \omega_3, \dots)) P^\infty(d(\omega_2, \omega_3, \dots)) \right) P(d\omega_1) \\
& = \int_X F(z) \pi_1^f(x, dz) + \sum_{n=1}^{\infty} \int_{\Omega^\infty} F(f^{n+1}(x, \omega_1, \omega_2, \dots)) P^\infty(d(\omega_1, \omega_2, \dots)) \\
& = \int_X F(z) \pi_1^f(x, dz) + \sum_{n=1}^{\infty} \int_X F(z) \pi_{n+1}^f(x, dz) = \varphi_0(x) - F(x).
\end{aligned}$$

□

Proposition 2.5. *If F is a real function on a metric space X and*

$$|F(x) - F(z)| \leq \beta \rho(x, z)^\alpha \quad \text{for } x, z \in X \quad (12)$$

with some constants $\alpha \in (0, 1)$, $\beta \in [0, \infty)$, then $F \in \mathcal{F}(X)$.

Proof. Fix $L \in (1, \frac{1}{\lambda})$, put

$$\vartheta = L^{-\frac{\alpha}{1-\alpha}}, \quad \theta = \vartheta^{\frac{1}{\alpha}},$$

and for every $n \in \mathbb{N}$ let A_n be a maximal for inclusion subset of X such that

$$\rho(x, z) \geq \theta^n \quad \text{for every pair of distinct points } x, z \text{ of } A_n.$$

By the maximality,

$$X = \bigcup_{z \in A_n} \{x \in X : \rho(x, z) < \theta^n\} \quad \text{for } n \in \mathbb{N}.$$

If $n \in \mathbb{N}$ and x, z are distinct points of A_n , then by (12),

$$|F(x) - F(z)| \leq \beta \rho(x, z)^{\alpha-1} \rho(x, z) \leq \beta \theta^{(\alpha-1)n} \rho(x, z) = \beta L^n \rho(x, z).$$

It follows from this, using Kirszbraun–McShane extension theorem [7, Theorem 6.1.1], that for every $n \in \mathbb{N}$ there exists an $F_n : X \rightarrow \mathbb{R}$ such that

$$F_n|_{A_n} = F|_{A_n} \quad \text{and} \quad |F_n(x) - F_n(z)| \leq \beta L^n \rho(x, z) \quad \text{for } x, z \in X.$$

If $n \in \mathbb{N}$ and $x \in X$, then there is a $z \in A_n$ such that $\rho(x, z) < \theta^n$, and

$$\begin{aligned} |F(x) - F_n(x)| &\leq |F(x) - F(z)| + |F_n(z) - F_n(x)| \\ &\leq \beta \rho(x, z)^\alpha + \beta L^n \rho(x, z) \\ &\leq \beta \theta^{\alpha n} + \beta L^n \theta^n = 2\beta \vartheta^n. \end{aligned}$$

□

Corollary 2.6. *Assume (H). If $F : X \rightarrow \mathbb{R}$ satisfies (12) with some constants $\alpha \in (0, 1)$, $\beta \in [0, \infty)$, then formula (6) defines a solution $\varphi : X \rightarrow \mathbb{R}$ of (1) such that*

$$|\varphi(x) - \varphi(z)| \leq \frac{\beta}{1 - \lambda^\alpha} \rho(x, z)^\alpha \quad \text{for } x, z \in X,$$

and if additionally (7) holds, then formula (8) defines a solution $\varphi_0 : X \rightarrow \mathbb{R}$ of (2) such that

$$|\varphi_0(x) - \varphi_0(z)| \leq \frac{\beta}{1 - \lambda^\alpha} \rho(x, z)^\alpha \quad \text{for } x, z \in X.$$

Proof. By Proposition 2.5 and Theorem 2.1 formula (6) defines a solution $\varphi : X \rightarrow \mathbb{R}$ of (1). Using (6), (11), (12), Jensen's inequality and (10) for every $x, z \in X$ we have

$$\begin{aligned}
|\varphi(x) - \varphi(z)| &\leq |F(x) - F(z)| \\
&\quad + \sum_{n=1}^{\infty} \int_{\Omega^{\infty}} |F(f^n(x, \omega)) - F(f^n(z, \omega))| P^{\infty}(d\omega) \\
&\leq \beta \rho(x, z)^{\alpha} + \sum_{n=1}^{\infty} \int_{\Omega^{\infty}} \beta \rho(f^n(x, \omega), f^n(z, \omega))^{\alpha} P^{\infty}(d\omega) \\
&\leq \beta \rho(x, z)^{\alpha} + \beta \sum_{n=1}^{\infty} \left(\int_{\Omega^{\infty}} \rho(f^n(x, \omega), f^n(z, \omega)) P^{\infty}(d\omega) \right)^{\alpha} \\
&\leq \beta \rho(x, z)^{\alpha} + \beta \sum_{n=1}^{\infty} (\lambda^n \rho(x, z))^{\alpha} = \frac{\beta}{1 - \lambda^{\alpha}} \rho(x, z)^{\alpha}.
\end{aligned}$$

For the second part we argue similarly. \square

Regarding the uniqueness of solutions, we have the following theorem.

Theorem 2.7. *Assume (H) and let $F : X \rightarrow \mathbb{R}$.*

(i) *If $\varphi_1, \varphi_2 \in \mathcal{F}(X)$ are solutions of (1), then $\varphi_1 = \varphi_2$.*

(ii) *If $\varphi_1, \varphi_2 \in \mathcal{F}(X)$ are solutions of (2), then $\varphi_1 - \varphi_2$ is a constant function.*

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{F}(X)$ and put $\varphi = \varphi_1 - \varphi_2$. Then $\varphi \in \mathcal{F}(X)$. Corresponding to φ choose a sequence $(F_n)_{n \in \mathbb{N}}$ of real functions on X and constants $\vartheta \in (0, 1)$, $L \in (0, \frac{1}{\lambda})$ and $\alpha, \beta \in (0, \infty)$ as in the definition of $\mathcal{F}(X)$.

If φ_1, φ_2 are solutions of (1), then φ solves (1) with $F = 0$, and, by induction,

$$\varphi(x) = (-1)^n \int_{\Omega^{\infty}} \varphi(f^n(x, \omega)) P^{\infty}(d\omega) \quad \text{for } x \in X, n \in \mathbb{N}.$$

If φ_1, φ_2 are solutions of (2), then φ solves (2) with $F = 0$, and

$$\varphi(x) = \int_{\Omega^{\infty}} \varphi(f^n(x, \omega)) P^{\infty}(d\omega) \quad \text{for } x \in X, n \in \mathbb{N}.$$

In both cases

$$|\varphi(x) - \varphi(z)| \leq \int_{\Omega^{\infty}} |\varphi(f^n(x, \omega)) - \varphi(f^n(z, \omega))| P^{\infty}(d\omega)$$

for $x, z \in X$, $n \in \mathbb{N}$. Moreover,

$$\begin{aligned}
|\varphi(x) - \varphi(z)| &\leq |\varphi(x) - F_n(x)| + |F_n(x) - F_n(z)| + |F_n(z) - \varphi(z)| \\
&\leq 2\alpha \vartheta^n + |F_n(x) - F_n(z)| \quad \text{for } x, z \in X, n \in \mathbb{N}.
\end{aligned}$$

Consequently, applying among others (10),

$$\begin{aligned} |\varphi(x) - \varphi(z)| &\leq 2\alpha\vartheta^n + \int_{\Omega^\infty} |F_n(f^n(x, \omega)) - F_n(f^n(z, \omega))| P^\infty(d\omega) \\ &\leq \beta L^n \lambda^n \rho(x, z) \quad \text{for } x, z \in X, \quad n \in \mathbb{N}, \end{aligned}$$

whence $\varphi(x) = \varphi(z)$ for $x, z \in X$, i.e., φ is a constant function. Noting that if a constant φ solves (1) with $F = 0$, then $\varphi = 0$, we end the proof. \square

We finish with a qualitative result.

Following [6] by Jens Peter Reus Christensen we say that a Borel subset B of an abelian Polish group G is a *Haar zero set* if there is a probability Borel measure μ on G such that $\mu(B + x) = 0$ for every $x \in G$. See also [8] where measurability in abelian Polish groups related to Christensen's Haar zero set is studied.

Assume

(H₀) (Ω, \mathcal{A}, P) is a probability space, (X, ρ) is a compact metric space, $f : X \times \Omega \rightarrow X$ is $\mathcal{B} \otimes \mathcal{A}$ -measurable and (3) holds with a $\lambda \in (0, 1)$.

Assuming (H₀) we have in particular (4):

$$\int_{\Omega} \rho(f(x, \omega), x) P(d\omega) \leq \text{diam}(X) \quad \text{for } x \in X.$$

Moreover one can consider the Banach space $C(X)$ of all continuous real functions on X with the uniform norm and its subspace C_f ,

$$C_f = \left\{ F \in C(X) : \int_X F(x) \pi^f(dx) = 0 \right\}.$$

Clearly C_f is a closed linear subspace of $C(X)$ and (see, e.g., [7, Corollary 11.2.5]) $C(X)$ is a separable Banach space. We have also the following lemma.

Lemma 2.8. *Assume (H₀). If $F : X \rightarrow \mathbb{R}$ is continuous, then so is the function*

$$x \mapsto \int_{\Omega} F(f(x, \omega)) P(d\omega), \quad x \in X.$$

Proof. Fix $\varepsilon \in (0, \infty)$ and choose $\delta \in (0, \infty)$ such that

$$|F(x) - F(z)| \leq \varepsilon \quad \text{for } x, z \in X \text{ with } \rho(x, z) \leq \delta.$$

Then, by (3), for all $x, z \in X$,

$$\begin{aligned}
& \left| \int_{\Omega} F(f(x, \omega)) P(d\omega) - \int_{\Omega} F(f(z, \omega)) P(d\omega) \right| \\
& \leq \int_{\Omega} |F(f(x, \omega)) - F(f(z, \omega))| P(d\omega) \\
& \leq \varepsilon + \int_{\{\omega \in \Omega : \rho(f(x, \omega), f(z, \omega)) > \delta\}} |F(f(x, \omega)) - F(f(z, \omega))| P(d\omega) \\
& \leq \varepsilon + 2\|F\| P(\{\omega \in \Omega : \rho(f(x, \omega), f(z, \omega)) > \delta\}) \\
& \leq \varepsilon + 2\|F\| \frac{1}{\delta} \int_{\Omega} \rho(f(x, \omega), f(z, \omega)) P(d\omega) \leq \varepsilon + \frac{2\lambda\|F\|}{\delta} \rho(x, z),
\end{aligned}$$

and therefore the discussed function is continuous. \square

Let

$$\mathcal{F}_1 = \{F \in C(X) : \text{equation (1) has a continuous solution } \varphi : X \rightarrow \mathbb{R}\},$$

$$\mathcal{F}_2 = \{F \in C_f : \text{equation (2) has a continuous solution } \varphi : X \rightarrow \mathbb{R}\}.$$

Theorem 2.9. *Under the assumptions (H₀):*

(i) \mathcal{F}_1 is a Borel and dense subset of $C(X)$, and if $\mathcal{F}_1 \neq C(X)$, then \mathcal{F}_1 is of first category in $C(X)$ and a Haar zero subset of $C(X)$.

(ii) \mathcal{F}_2 is a Borel and dense subset of C_f , and if $\mathcal{F}_2 \neq C_f$, then \mathcal{F}_2 is of first category in C_f and a Haar zero subset of C_f .

Proof. By Lemma 2.8 the formulas

$$T_1(\varphi)(x) = \varphi(x) + \int_{\Omega} \varphi(f(x, \omega)) P(d\omega),$$

$$T_2(\varphi)(x) = \varphi(x) - \int_{\Omega} \varphi(f(x, \omega)) P(d\omega) \quad \text{for } \varphi \in C(X) \text{ and } x \in X,$$

define self-mappings T_1, T_2 of $C(X)$. Clearly, these operators are linear and continuous. Moreover,

$$T_1(C(X)) = \mathcal{F}_1.$$

Furthermore, for every $F \in T_2(C(X))$ Eq. (2) has a continuous solution $\varphi : X \rightarrow \mathbb{R}$. Hence (II) gives $T_2(C(X)) \subset C_f$, and

$$T_2(C(X)) = \mathcal{F}_2.$$

Applying now [1, Lemma] we see that \mathcal{F}_1 is a Borel subset of $C(X)$, and if $\mathcal{F}_1 \neq C(X)$, then \mathcal{F}_1 is of first category in $C(X)$ and a Haar zero subset of $C(X)$, and \mathcal{F}_2 is a Borel subset of C_f , and if $\mathcal{F}_2 \neq C_f$, then \mathcal{F}_2 is of first category in C_f and a Haar zero subset of C_f .

Since by (I) the set

$$\{F \in C(X) : F \text{ is Lipschitz}\}$$

is contained in \mathcal{F}_1 and (see [7, Theorem 11.2.4]) dense in $C(X)$, the set \mathcal{F}_1 is dense in $C(X)$.

To show that \mathcal{F}_2 is dense in C_f , fix $F \in C_f$ and $\varepsilon \in (0, \infty)$. Choose a Lipschitz $F_1 : X \rightarrow \mathbb{R}$ so that $\|F - F_1\| < \frac{\varepsilon}{2}$. According to (III), $F_1 - \int_X F_1 d\pi^f \in \mathcal{F}_2$. Moreover,

$$\left\| F - \left(F_1 - \int_X F_1 d\pi^f \right) \right\| \leq \|F - F_1\| + \left| \int_X F_1 d\pi^f - \int_X F d\pi^f \right| < \varepsilon.$$

□

Remark 2.10. It is possible that (H_0) holds and $\mathcal{F}_1 = C(X)$, $\mathcal{F}_2 = C_f$.

To see it consider an \mathcal{A} -measurable $\xi : \Omega \rightarrow X$ and let $f(x, \omega) = \xi(\omega)$ for $(x, \omega) \in X \times \Omega$. Then $f^n(x, \omega) = \xi(\omega_n)$ for $(x, \omega) \in X \times \Omega^\infty$, so $\pi_n^f(x, B) = P(\xi \in B) = \pi^f(B)$ for $n \in \mathbb{N}$, $x \in X$, $B \in \mathcal{B}$, and $\int_X F d\pi^f = \int_\Omega F \circ \xi dP$ for $F \in C(X)$. Consequently for every $F \in C(X)$ the function $F - \frac{1}{2} \int_X F d\pi^f$ solves (1), and every $F \in C_f$ solves (2).

Acknowledgements

The research was supported by the Institute of Mathematics of the University of Silesia (Iterative Functional Equations and Real Analysis program).

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Baron, K.: A remark on linear functional equations in the indeterminate case. *Glasnik Mat.* **20**(40), 373–376 (1985)
- [2] Baron, K.: On the convergence in law of iterates of random-valued functions. *Aust. J. Math. Anal. Appl.* **6**, no. 1, Art. 3 (2009)
- [3] Baron, K.: Weak limit of iterates of some random-valued functions and its application. *Aequ. Math.* **94**, 415–425; 427 (Correction) (2020)
- [4] Baron, K.: Around the weak limit of iterates of some random-valued functions. *Ann. Univ. Budapest. Sect. Comput.* **51**, 31–37 (2020)

- [5] Baron, K., Kapica, R., Morawiec, J.: On Lipschitzian solutions to an inhomogeneous linear iterative equation. *Aequ. Math.* **90**, 77–85 (2016)
- [6] Christensen, J.P.R.: On sets of Haar measure zero in abelian Polish groups. *Isr. J. Math.* **13**, 255–260 (1972)
- [7] Dudley, R.M.: *Real Analysis and Probability*. Cambridge Studies in Advanced Mathematics, vol. 74. Cambridge University Press, Cambridge (2002)
- [8] Fischer, P., Ślódkowski, Z.: Christensen zero sets and measurable convex functions. *Proc. Amer. Math. Soc.* **79**, 449–453 (1980)
- [9] Jarczyk, W.: On a set of functional equations having continuous solutions. *Glasnik Mat.* **17**(37), 59–64 (1982)
- [10] Kapica, R.: Sequences of iterates of random-valued vector functions and solutions of related equations. *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II* **213**, 113–118 (2005)
- [11] Kapica, R.: The geometric rate of convergence of random iteration in the Hutchinson distance. *Aequ. Math.* **93**, 149–160 (2019)
- [12] Kuczma, M.: *Functional equations in a single variable*. Monografie Matematyczne, vol. 48. PWN–Polish Scientific Publishers, Warszawa (1968)
- [13] Kuczma, M., Choczewski, B., Ger, R.: *Iterative Functional Equations*. Encyclopedia of Mathematics and Its Applications, vol. 32. Cambridge University Press, Cambridge (1990)

Karol Baron
Institute of Mathematics
University of Silesia
Bankowa 14
40-007 Katowice
Poland
e-mail: baron@us.edu.pl

Received: October 8, 2020

Revised: February 27, 2021

Accepted: March 3, 2021

Terms and Conditions

Springer Nature journal content, brought to you courtesy of Springer Nature Customer Service Center GmbH (“Springer Nature”).

Springer Nature supports a reasonable amount of sharing of research papers by authors, subscribers and authorised users (“Users”), for small-scale personal, non-commercial use provided that all copyright, trade and service marks and other proprietary notices are maintained. By accessing, sharing, receiving or otherwise using the Springer Nature journal content you agree to these terms of use (“Terms”). For these purposes, Springer Nature considers academic use (by researchers and students) to be non-commercial.

These Terms are supplementary and will apply in addition to any applicable website terms and conditions, a relevant site licence or a personal subscription. These Terms will prevail over any conflict or ambiguity with regards to the relevant terms, a site licence or a personal subscription (to the extent of the conflict or ambiguity only). For Creative Commons-licensed articles, the terms of the Creative Commons license used will apply.

We collect and use personal data to provide access to the Springer Nature journal content. We may also use these personal data internally within ResearchGate and Springer Nature and as agreed share it, in an anonymised way, for purposes of tracking, analysis and reporting. We will not otherwise disclose your personal data outside the ResearchGate or the Springer Nature group of companies unless we have your permission as detailed in the Privacy Policy.

While Users may use the Springer Nature journal content for small scale, personal non-commercial use, it is important to note that Users may not:

1. use such content for the purpose of providing other users with access on a regular or large scale basis or as a means to circumvent access control;
2. use such content where to do so would be considered a criminal or statutory offence in any jurisdiction, or gives rise to civil liability, or is otherwise unlawful;
3. falsely or misleadingly imply or suggest endorsement, approval, sponsorship, or association unless explicitly agreed to by Springer Nature in writing;
4. use bots or other automated methods to access the content or redirect messages
5. override any security feature or exclusionary protocol; or
6. share the content in order to create substitute for Springer Nature products or services or a systematic database of Springer Nature journal content.

In line with the restriction against commercial use, Springer Nature does not permit the creation of a product or service that creates revenue, royalties, rent or income from our content or its inclusion as part of a paid for service or for other commercial gain. Springer Nature journal content cannot be used for inter-library loans and librarians may not upload Springer Nature journal content on a large scale into their, or any other, institutional repository.

These terms of use are reviewed regularly and may be amended at any time. Springer Nature is not obligated to publish any information or content on this website and may remove it or features or functionality at our sole discretion, at any time with or without notice. Springer Nature may revoke this licence to you at any time and remove access to any copies of the Springer Nature journal content which have been saved.

To the fullest extent permitted by law, Springer Nature makes no warranties, representations or guarantees to Users, either express or implied with respect to the Springer nature journal content and all parties disclaim and waive any implied warranties or warranties imposed by law, including merchantability or fitness for any particular purpose.

Please note that these rights do not automatically extend to content, data or other material published by Springer Nature that may be licensed from third parties.

If you would like to use or distribute our Springer Nature journal content to a wider audience or on a regular basis or in any other manner not expressly permitted by these Terms, please contact Springer Nature at

onlineservice@springernature.com