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Continuous solutions to two iterative functional equations

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Dedicated to Professor Ludwig Reich on his 80th birthday.

Abstract. Based on iteration of random-valued functions we study the problem of solvability in the class of continuous and Hölder continuous functions φ of the equations

$$\varphi(x) = F(x) - \int_\Omega \varphi\big(f(x,\omega)\big)P(d\omega),$$
$$\varphi(x) = F(x) + \int_\Omega \varphi\big(f(x,\omega)\big)P(d\omega),$$

where P is a probability measure on a σ-algebra of subsets of Ω.

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1. Introduction

Fix a probability space $(\Omega, \mathcal{A}, P)$, a complete and separable metric space (X, ρ) with the σ-algebra $\mathcal{B}$ of all its Borel subsets, and a $\mathcal{B} \otimes \mathcal{A}$-measurable function $f : X \times \Omega \to X$.

We continue the research of continuous solutions $\varphi : X \to \mathbb{R}$ of the equations

$$\varphi(x) = F(x) - \int_\Omega \varphi\big(f(x,\omega)\big)P(d\omega), \tag{1}$$

$$\varphi(x) = F(x) + \int_\Omega \varphi\big(f(x,\omega)\big)P(d\omega). \tag{2}$$

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We refer mainly to [2,5]. Like in these papers we focus on the iteration of random-valued functions:

$$f^0(x,\omega_1,\omega_2,\dots)=x, \quad f^n(x,\omega_1,\omega_2,\dots)=f\big(f^{n-1}(x,\omega_1,\omega_2,\dots),\omega_n\big)$$

for $n\in\mathbb{N}$, $x\in X$ and $(\omega_1,\omega_2,\dots)$ from Ω^∞ defined as $\Omega^{\mathbb{N}}$. Note that for $n\in\mathbb{N}$ the nth iterate f^n mapping $X\times\Omega^\infty$ into X is $\mathcal{B}\otimes\mathcal{A}_n$-measurable, where $\mathcal{A}_n$ denotes the σ-algebra of all sets of the form

$$\{(\omega_1,\omega_2,\dots)\in\Omega^\infty:(\omega_1,\dots,\omega_n)\in A\}$$

with A from the product σ-algebra $\mathcal{A}^n$. (See [13, section 1.4], [10].)

Let $\pi_n^f(x,\cdot)$ denote the distribution of $f^n(x,\cdot)$, i.e.,

$$\pi_n^f(x,B)=P^\infty\big(f^n(x,\cdot)\in B\big) \quad \text{for } n\in\mathbb{N}\cup\{0\},\ x\in X \text{ and } B\in\mathcal{B}.$$

If

$$\int_\Omega \rho\big(f(x,\omega),f(z,\omega)\big)P(d\omega)\le\lambda\rho(x,z) \quad \text{for } x,z\in X \tag{3}$$

with a $\lambda\in(0,1)$, and

$$\int_\Omega \rho\big(f(x,\omega),x\big)P(d\omega)<\infty \quad \text{for } x\in X, \tag{4}$$

then (see [2, Theorem 3.1]) there exists a probability Borel measure π^f on X such that for every $x\in X$ the sequence $(\pi_n^f(x,\cdot))_{n\in\mathbb{N}}$ converges weakly to π^f, and (see [11, Corollary 5.6 and Lemma 3.1], also [3, Lemma 2.2]) for every non-expansive $u:X\to\mathbb{R}$ the inequality

$$\left|\int_X u(z)\pi_n^f(x,dz)-\int_X u(z)\pi^f(dz)\right|\le\frac{\lambda^n}{1-\lambda}\int_\Omega\rho\big(f(x,\omega),x\big)P(d\omega) \tag{5}$$

holds for $x\in X$ and $n\in\mathbb{N}$.

This limit distribution π^f plays an important role in solving (1) and (2), see [5, Theorem 3.1], [2, Corollary 4.1], [3, Theorem 2.1]. In particular:

(I) If $F:X\to\mathbb{R}$ is continuous and bounded, then any continuous and bounded solution $\varphi:X\to\mathbb{R}$ of (1) has the form

$$\begin{aligned}\varphi(x)&=F(x)-\frac{1}{2}\int_X F(z)\pi^f(dz)\\&\quad+\sum_{n=1}^\infty(-1)^n\left(\int_X F(z)\pi_n^f(x,dz)-\int_X F(z)\pi^f(dz)\right) \quad \text{for } x\in X;\end{aligned} \tag{6}$$

if additionally F is Lipschitz, then (6) defines a Lipschitz solution $\varphi:X\to\mathbb{R}$ of (1).

(II) If $F:X\to\mathbb{R}$ is continuous and bounded and (2) has a continuous and bounded solution $\varphi:X\to\mathbb{R}$, then

$$\int_X F(x)\pi^f(dx)=0, \tag{7}$$

and any such solution has the form

$$\varphi(x) = c + F(x) + \sum_{n=1}^{\infty} \int_X F(z)\pi_n^f(x, dz) \quad \text{for } x \in X$$

with a real constant c.

(III) If $F : X \to \mathbb{R}$ is Lipschitz, then it is integrable for π^f and (2) has a Lipschitz solution $\varphi : X \to \mathbb{R}$ if and only if (7) holds.

The limit distribution π^f and facts cited above will be used in the main part of the paper. A characterization of this limit for some special random-valued functions in Hilbert spaces have been given by [3, Theorem 3.1] and, in Banach spaces, by [4, Theorem 2.1].

Actually we do not have a sufficiently satisfactory theorem to guarantee the existence of continuous solutions to the equations considered. An explanation of this situation is given in the paper [9] by Witold Jarczyk (see also [13, Note 3.8.4]). Namely, in the case where Ω is a singleton and X is a compact real interval, for the appropriate f the set of continuous $F : X \to \mathbb{R}$ such that the equation has a continuous solution is small in the sense of Baire category. It is also small from the measure point of view (see [1]). We will go also in this direction but, above all, we are looking for conditions under which Eqs. (1) and (2) have continuous and Hölder continuous solutions $\varphi : X \to \mathbb{R}$. In the case where Ω is a singleton, see [12, Chapter II, §7].

2. Results

We will consider Eqs. (1) and (2) assuming the following hypothesis (H).

(H) $(\Omega, \mathcal{A}, P)$ is a probability space, (X, ρ) is a complete and separable metric space, $f : X \times \Omega \to X$ is $\mathcal{B} \otimes \mathcal{A}$-measurable, (3) holds with a $\lambda \in (0, 1)$ and (4) is satisfied.

We regard λ as fixed in $(0, 1)$, and for any metric space X we define $\mathcal{F}(X)$ as the set of all continuous functions $F : X \to \mathbb{R}$ such that there are a sequence $(F_n)_{n\in\mathbb{N}}$ of real functions on X and constants $\vartheta \in (0, 1)$, $L \in (0, \frac{1}{\lambda})$ and $\alpha, \beta \in (0, \infty)$ such that

$$|F(x) - F_n(x)| \leq \alpha\vartheta^n \quad \text{for } x \in X, \; n \in \mathbb{N},$$

and

$$|F_n(x) - F_n(z)| \leq \beta L^n \rho(x, z) \quad \text{for } x, z \in X, \; n \in \mathbb{N}.$$

Clearly any real Lipschitz function defined on X belongs to $\mathcal{F}(X)$.

Theorem 2.1. *Assume* (H). *If $F \in \mathcal{F}(X)$, then formula* (6) *defines a continuous solution $\varphi : X \to \mathbb{R}$ of* (1), *and if additionally* (7) *holds, then the formula*

$$\varphi_0(x) = F(x) + \sum_{n=1}^{\infty} \int_X F(z)\pi_n^f(x, dz) \quad \textit{for } x \in X \tag{8}$$

defines a continuous solution $\varphi_0 : X \to \mathbb{R}$ *of* (2)*.*

The proof will be based on three lemmas. In each of them we assume (H).

Lemma 2.2. *If* $F \in \mathcal{F}(X)$*, then the integrals*

$$\int_\Omega \big|F\big(f(x,\omega)\big)\big|P(d\omega) \quad \text{for } x \in X, \qquad \int_X |F(z)|\pi^f(dz)$$

are finite, and the function

$$x \mapsto \int_\Omega F\big(f(x,\omega)\big)P(d\omega), \quad x \in X, \tag{9}$$

is continuous.

Proof. Corresponding to F choose a sequence $(F_n)_{n\in\mathbb{N}}$ of real functions on X and constants $\vartheta \in (0,1)$, $L \in (0, \frac{1}{\lambda})$ and $\alpha, \beta \in (0,\infty)$ as in the definition of $\mathcal{F}(X)$. Then

$$\int_\Omega \big|F\big(f(x,\omega)\big)\big|P(d\omega) \le \alpha\vartheta + \beta L \int_\Omega \rho\big(f(x,\omega),x\big)P(d\omega) + |F_1(x)|$$

for $x \in X$, and

$$\int_X |F(z)|\pi^f(dz) \le \alpha\vartheta + \int_X |F_1(z)|\pi^f(dz),$$

see also (III). Moreover, for every $n \in \mathbb{N}$ the function

$$x \mapsto \int_\Omega F_n\big(f(x,\omega)\big)P(d\omega), \quad x \in X,$$

is Lipschitz:

$$\left|\int_\Omega F_n\big(f(x,\omega)\big)P(d\omega) - \int_\Omega F_n\big(f(z,\omega)\big)P(d\omega)\right| \le \beta L^n \lambda\rho(x,z)$$

for $x, z \in X$, and therefore function (9), as their uniform limit, is continuous. □

Lemma 2.3. *If* $F \in \mathcal{F}(X)$*, then*

$$\int_X |F(z)|\pi_n^f(x,dz) < \infty \quad \text{for } x \in X \text{ and } n \in \mathbb{N},$$

and for every $n \in \mathbb{N}$ *the function*

$$x \mapsto \int_X F(z)\pi_n^f(x,dz), \quad x \in X,$$

is continuous.

Proof. By induction, (3) and (4),

$$\int_{\Omega^\infty} \rho\big(f^n(x,\omega), f^n(z,\omega)\big) P^\infty(d\omega) \le \lambda^n \rho(x,z) \quad \text{for } x, z \in X \text{ and } n \in \mathbb{N} \tag{10}$$

and

$$\int_{\Omega^\infty} \rho\big(f^n(x,\omega), x\big) P^\infty(d\omega) < \infty \quad \text{for } x \in X \text{ and } n \in \mathbb{N}.$$

Since

$$\int_X F(z)\pi_n^f(x,dz) = \int_{\Omega^\infty} F\big(f^n(x,\omega)\big) P^\infty(d\omega) \quad \text{for } x \in X \text{ and } n \in \mathbb{N}, \tag{11}$$

an application of Lemma 2.2 with f replaced by f^n, $n \in \mathbb{N}$, finishes the proof.
□

Lemma 2.4. *If $F \in \mathcal{F}(X)$, then there are constants $\theta \in (0,1)$ and $M \in (0,\infty)$ such that*

$$\left| \int_X F(z)\pi_n^f(x,dz) - \int_X F(z)\pi^f(dz) \right| \le M\theta^n \Big(1 + \rho(x,x_0) + \int_\Omega \rho\big(f(x_0,\omega), x_0\big) P(d\omega)\Big)$$

for $x, x_0 \in X$ and $n \in \mathbb{N}$.

Proof. Corresponding to F choose a sequence $(F_n)_{n\in\mathbb{N}}$ of real functions on X and constants $\vartheta \in (0,1)$, $L \in (0, \frac{1}{\lambda})$ and $\alpha, \beta \in (0,\infty)$ as in the definition of $\mathcal{F}(X)$, and put

$$\theta = \max\{\vartheta, \lambda L\}, \quad M = 2\max\left\{\alpha, \frac{\beta}{1-\lambda}\right\}.$$

Then $\theta \in (0,1)$, and by Lemmas 2.3 and 2.2, (5) with $u = \frac{F_n}{\beta L^n}$ and (3) for every $x, x_0 \in X$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\left|\int_X F(z)\pi_n^f(x,dz) - \int_X F(z)\pi^f(dz)\right| \\
&\quad \le \left|\int_X F(z)\pi_n^f(x,dz) - \int_X F_n(z)\pi_n^f(x,dz)\right| \\
&\qquad + \left|\int_X F_n(z)\pi_n^f(x,dz) - \int_X F_n(z)\pi^f(dz)\right| \\
&\qquad + \left|\int_X F_n(z)\pi^f(dz) - \int_X F(z)\pi^f(dz)\right| \\
&\quad \le \int_X |F(z)-F_n(z)|\pi_n^f(x,dz) + \beta L^n \frac{\lambda^n}{1-\lambda}\int_\Omega \rho\big(f(x,\omega),x\big)P(d\omega) \\
&\qquad + \int_X |F_n(z)-F(z)|\pi^f(dz) \le 2\alpha\vartheta^n + \beta L^n\frac{\lambda^n}{1-\lambda}\int_\Omega \rho\big(f(x,\omega),x\big)P(d\omega) \\
&\quad \le 2\alpha\theta^n + \frac{\beta}{1-\lambda}\theta^n\left(\lambda\rho(x,x_0) + \int_\Omega \rho\big(f(x_0,\omega),x_0\big)P(d\omega) + \rho(x,x_0)\right) \\
&\quad \le M\theta^n\left(1+\rho(x,x_0) + \int_\Omega \rho\big(f(x_0,\omega),x_0\big)P(d\omega)\right).
\end{aligned}$$

□

Proof of Theorem 2.1. It follows from Lemmas 2.2–2.4 that formula (6) defines a continuous function $\varphi : X \to \mathbb{R}$ and arguing like in the proof of Theorem 3.1(ii) of [5] (see also the calculations below) we show that it solves (1).

Assume now that also (7) holds. Then it follows from Lemmas 2.3 and 2.4 that formula (8) defines a continuous function $\varphi_0 : X \to \mathbb{R}$. Applying (11), Lemma 2.4, the Lebesgue dominated convergence theorem and the Fubini theorem we observe that for every $x \in X$ the function $\varphi_0 \circ f(x,\cdot)$ is integrable for P and

$$\begin{aligned}
&\int_\Omega \varphi_0\big(f(x,\omega)\big)P(d\omega) = \int_\Omega F\big(f(x,\omega)\big)P(d\omega) \\
&\quad + \int_\Omega \sum_{n=1}^\infty \left(\int_X F(z)\pi_n^f\big(f(x,\omega),dz\big)\right)P(d\omega) = \int_X F(z)\pi_1^f(x,dz) \\
&\quad + \sum_{n=1}^\infty \int_\Omega \left(\int_{\Omega^\infty} F\big(f^n\big(f(x,\omega_1),\omega_2,\omega_3,\ldots\big)\big)P^\infty\big(d(\omega_2,\omega_3,\ldots)\big)\right)P(d\omega_1) \\
&= \int_X F(z)\pi_1^f(x,dz) + \sum_{n=1}^\infty \int_{\Omega^\infty} F\big(f^{n+1}(x,\omega_1,\omega_2,\ldots)\big)P^\infty\big(d(\omega_1,\omega_2,\ldots)\big) \\
&= \int_X F(z)\pi_1^f(x,dz) + \sum_{n=1}^\infty \int_X F(z)\pi_{n+1}^f(x,dz) = \varphi_0(x) - F(x).
\end{aligned}$$

□

Proposition 2.5. *If F is a real function on a metric space X and*

$$|F(x) - F(z)| \le \beta\rho(x,z)^{\alpha} \quad \text{for } x, z \in X \tag{12}$$

with some constants $\alpha \in (0,1)$, $\beta \in [0,\infty)$, then $F \in \mathcal{F}(X)$.

Proof. Fix $L \in (1, \frac{1}{\lambda})$, put

$$\vartheta = L^{-\frac{\alpha}{1-\alpha}}, \quad \theta = \vartheta^{\frac{1}{\alpha}},$$

and for every $n \in \mathbb{N}$ let A_n be a maximal for inclusion subset of X such that

$$\rho(x,z) \ge \theta^n \quad \text{for every pair of distinct points } x, z \text{ of } A_n.$$

By the maximality,

$$X = \bigcup_{z \in A_n} \{x \in X : \rho(x,z) < \theta^n\} \quad \text{for } n \in \mathbb{N}.$$

If $n \in \mathbb{N}$ and x, z are distinct points of A_n, then by (12),

$$|F(x) - F(z)| \le \beta\rho(x,z)^{\alpha-1}\rho(x,z) \le \beta\theta^{(\alpha-1)n}\rho(x,z) = \beta L^n \rho(x,z).$$

It follows from this, using Kirszbraun–McShane extension theorem [7, Theorem 6.1.1], that for every $n \in \mathbb{N}$ there exists an $F_n : X \to \mathbb{R}$ such that

$$F_n|_{A_n} = F|_{A_n} \quad \text{and} \quad |F_n(x) - F_n(z)| \le \beta L^n \rho(x,z) \quad \text{for } x, z \in X.$$

If $n \in \mathbb{N}$ and $x \in X$, then there is a $z \in A_n$ such that $\rho(x,z) < \theta^n$, and

$$\begin{aligned} |F(x) - F_n(x)| &\le |F(x) - F(z)| + |F_n(z) - F_n(x)| \\ &\le \beta\rho(x,z)^{\alpha} + \beta L^n \rho(x,z) \\ &\le \beta\theta^{\alpha n} + \beta L^n \theta^n = 2\beta\vartheta^n. \end{aligned}$$

□

Corollary 2.6. *Assume* (H)*. If $F : X \to \mathbb{R}$ satisfies* (12) *with some constants $\alpha \in (0,1)$, $\beta \in [0,\infty)$, then formula* (6) *defines a solution $\varphi : X \to \mathbb{R}$ of* (1) *such that*

$$|\varphi(x) - \varphi(z)| \le \frac{\beta}{1-\lambda^{\alpha}}\rho(x,z)^{\alpha} \quad \text{for } x, z \in X,$$

and if additionally (7) *holds, then formula* (8) *defines a solution $\varphi_0 : X \to \mathbb{R}$ of* (2) *such that*

$$|\varphi_0(x) - \varphi_0(z)| \le \frac{\beta}{1-\lambda^{\alpha}}\rho(x,z)^{\alpha} \quad \text{for } x, z \in X.$$

Proof. By Proposition 2.5 and Theorem 2.1 formula (6) defines a solution $\varphi : X \to \mathbb{R}$ of (1). Using (6), (11), (12), Jensen's inequality and (10) for every $x, z \in X$ we have

$$\begin{aligned}|\varphi(x) - \varphi(z)| &\le |F(x) - F(z)| \\ &\quad + \sum_{n=1}^{\infty} \int_{\Omega^\infty} \left|F\big(f^n(x,\omega)\big) - F\big(f^n(z,\omega)\big)\right| P^\infty(d\omega) \\ &\le \beta\rho(x,z)^\alpha + \sum_{n=1}^{\infty} \int_{\Omega^\infty} \beta\rho\big(f^n(x,\omega), f^n(z,\omega)\big)^\alpha P^\infty(d\omega) \\ &\le \beta\rho(x,z)^\alpha + \beta\sum_{n=1}^{\infty} \left(\int_{\Omega^\infty} \rho\big(f^n(x,\omega), f^n(z,\omega)\big) P^\infty(d\omega)\right)^\alpha \\ &\le \beta\rho(x,z)^\alpha + \beta\sum_{n=1}^{\infty} \big(\lambda^n\rho(x,z)\big)^\alpha = \frac{\beta}{1-\lambda^\alpha}\rho(x,z)^\alpha.\end{aligned}$$

For the second part we argue similarly. □

Regarding the uniqueness of solutions, we have the following theorem.

Theorem 2.7. *Assume* (H) *and let* $F : X \to \mathbb{R}$.

(i) *If* $\varphi_1, \varphi_2 \in \mathcal{F}(X)$ *are solutions of* (1)*, then* $\varphi_1 = \varphi_2$.

(ii) *If* $\varphi_1, \varphi_2 \in \mathcal{F}(X)$ *are solutions of* (2)*, then* $\varphi_1 - \varphi_2$ *is a constant function.*

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{F}(X)$ and put $\varphi = \varphi_1 - \varphi_2$. Then $\varphi \in \mathcal{F}(X)$. Corresponding to φ choose a sequence $(F_n)_{n\in\mathbb{N}}$ of real functions on X and constants $\vartheta \in (0,1)$, $L \in (0, \frac{1}{\lambda})$ and $\alpha, \beta \in (0,\infty)$ as in the definition of $\mathcal{F}(X)$.

If φ_1, φ_2 are solutions of (1), then φ solves (1) with $F = 0$, and, by induction,

$$\varphi(x) = (-1)^n \int_{\Omega^\infty} \varphi\big(f^n(x,\omega)\big) P^\infty(d\omega) \quad \text{for } x \in X,\ n \in \mathbb{N}.$$

If φ_1, φ_2 are solutions of (2), then φ solves (2) with $F = 0$, and

$$\varphi(x) = \int_{\Omega^\infty} \varphi\big(f^n(x,\omega)\big) P^\infty(d\omega) \quad \text{for } x \in X,\ n \in \mathbb{N}.$$

In both cases

$$|\varphi(x) - \varphi(z)| \le \int_{\Omega^\infty} \left|\varphi\big(f^n(x,\omega)\big) - \varphi\big(f^n(z,\omega)\big)\right| P^\infty(d\omega)$$

for $x, z \in X,\ n \in \mathbb{N}$. Moreover,

$$\begin{aligned}|\varphi(x) - \varphi(z)| &\le |\varphi(x) - F_n(x)| + |F_n(x) - F_n(z)| + |F_n(z) - \varphi(z)| \\ &\le 2\alpha\vartheta^n + |F_n(x) - F_n(z)| \quad \text{for } x, z \in X,\ n \in \mathbb{N}.\end{aligned}$$

Consequently, applying among others (10),

$$
\begin{aligned}
|\varphi(x)-\varphi(z)| \le 2\alpha\vartheta^n &+ \int_{\Omega^\infty} \big|F_n\big(f^n(x,\omega)\big) - F_n\big(f^n(z,\omega)\big)\big| P^\infty(d\omega) \\
&\le \beta L^n \lambda^n \rho(x,z) \quad \text{for } x,z \in X,\ n \in \mathbb{N},
\end{aligned}
$$

whence $\varphi(x) = \varphi(z)$ for $x, z \in X$, i.e., φ is a constant function. Noting that if a constant φ solves (1) with $F = 0$, then $\varphi = 0$, we end the proof. □

We finish with a qualitative result.

Following [6] by Jens Peter Reus Christensen we say that a Borel subset B of an abelian Polish group G is a *Haar zero set* if there is a probability Borel measure μ on G such that $\mu(B + x) = 0$ for every $x \in G$. See also [8] where measurability in abelian Polish groups related to Christensen's Haar zero set is studied.

Assume

(H_0) $(\Omega, \mathcal{A}, P)$ is a probability space, (X, ρ) is a compact metric space, $f : X \times \Omega \to X$ is $\mathcal{B} \otimes \mathcal{A}$-measurable and (3) holds with a $\lambda \in (0, 1)$.

Assuming (H_0) we have in particular (4):

$$
\int_\Omega \rho\big(f(x,\omega), x\big) P(d\omega) \le \operatorname{diam}(X) \quad \text{for } x \in X.
$$

Moreover one can consider the Banach space $C(X)$ of all continuous real functions on X with the uniform norm and its subspace C_f,

$$
C_f = \left\{ F \in C(X) : \int_X F(x)\pi^f(dx) = 0 \right\}.
$$

Clearly C_f is a closed linear subspace of $C(X)$ and (see, e.g., [7, Corollary 11.2.5]) $C(X)$ is a separable Banach space. We have also the following lemma.

Lemma 2.8. *Assume* (H_0)*. If* $F : X \to \mathbb{R}$ *is continuous, then so is the function*

$$
x \mapsto \int_\Omega F\big(f(x,\omega)\big) P(d\omega), \quad x \in X.
$$

Proof. Fix $\varepsilon \in (0, \infty)$ and choose $\delta \in (0, \infty)$ such that

$$
|F(x) - F(z)| \le \varepsilon \quad \text{for } x, z \in X \text{ with } \rho(x,z) \le \delta.
$$

Then, by (3), for all $x, z \in X$,

$$\begin{aligned}&\left|\int_\Omega F\big(f(x,\omega)\big)P(d\omega) - \int_\Omega F\big(f(z,\omega)\big)P(d\omega)\right|\\&\quad\le \int_\Omega \big|F\big(f(x,\omega)\big) - F\big(f(z,\omega)\big)\big|P(d\omega)\\&\quad\le \varepsilon + \int_{\{\omega\in\Omega:\rho(f(x,\omega),f(z,\omega))>\delta\}} \big|F\big(f(x,\omega)\big) - F\big(f(z,\omega)\big)\big|P(d\omega)\\&\quad\le \varepsilon + 2\|F\|P\big(\{\omega\in\Omega : \rho\big(f(x,\omega),f(z,\omega)\big)>\delta\}\big)\\&\quad\le \varepsilon + 2\|F\|\frac{1}{\delta}\int_\Omega \rho\big(f(x,\omega),f(z,\omega)\big)P(d\omega) \le \varepsilon + \frac{2\lambda\|F\|}{\delta}\rho(x,z),\end{aligned}$$

and therefore the discussed function is continuous. □

Let

$$\mathcal{F}_1 = \{F \in C(X) :\ \text{equation (1) has a continuous solution } \varphi : X \to \mathbb{R}\},$$
$$\mathcal{F}_2 = \{F \in C_f :\ \text{equation (2) has a continuous solution } \varphi : X \to \mathbb{R}\}.$$

Theorem 2.9. *Under the assumptions* (H_0):

(i) *$\mathcal{F}_1$ is a Borel and dense subset of $C(X)$, and if $\mathcal{F}_1 \neq C(X)$, then $\mathcal{F}_1$ is of first category in $C(X)$ and a Haar zero subset of $C(X)$.*

(ii) *$\mathcal{F}_2$ is a Borel and dense subset of C_f, and if $\mathcal{F}_2 \neq C_f$, then $\mathcal{F}_2$ is of first category in C_f and a Haar zero subset of C_f.*

Proof. By Lemma 2.8 the formulas

$$T_1(\varphi)(x) = \varphi(x) + \int_\Omega \varphi\big(f(x,\omega)\big)P(d\omega),$$
$$T_2(\varphi)(x) = \varphi(x) - \int_\Omega \varphi\big(f(x,\omega)\big)P(d\omega) \quad \text{for } \varphi \in C(X) \text{ and } x \in X,$$

define self-mappings T_1, T_2 of $C(X)$. Clearly, these operators are linear and continuous. Moreover,

$$T_1(C(X)) = \mathcal{F}_1.$$

Furthermore, for every $F \in T_2(C(X))$ Eq. (2) has a continuous solution $\varphi : X \to \mathbb{R}$. Hence (II) gives $T_2(C(X)) \subset C_f$, and

$$T_2(C(X)) = \mathcal{F}_2.$$

Applying now [1, Lemma] we see that $\mathcal{F}_1$ is a Borel subset of $C(X)$, and if $\mathcal{F}_1 \neq C(X)$, then $\mathcal{F}_1$ is of first category in $C(X)$ and a Haar zero subset of $C(X)$, and $\mathcal{F}_2$ is a Borel subset of C_f, and if $\mathcal{F}_2 \neq C_f$, then $\mathcal{F}_2$ is of first category in C_f and a Haar zero subset of C_f.

Since by (I) the set

$$\{F \in C(X) : F \text{ is Lipschitz}\}$$

is contained in $\mathcal{F}_1$ and (see [7, Theorem 11.2.4]) dense in $C(X)$, the set $\mathcal{F}_1$ is dense in $C(X)$.

To show that $\mathcal{F}_2$ is dense in C_f, fix $F \in C_f$ and $\varepsilon \in (0,\infty)$. Choose a Lipschitz $F_1 : X \to \mathbb{R}$ so that $\|F - F_1\| < \frac{\varepsilon}{2}$. According to (III), $F_1 - \int_X F_1 d\pi^f \in \mathcal{F}_2$. Moreover,

$$\left\| F - \left(F_1 - \int_X F_1 d\pi^f\right)\right\| \leq \|F - F_1\| + \left|\int_X F_1 d\pi^f - \int_X F d\pi^f\right| < \varepsilon.$$

□

Remark 2.10. It is possible that (H_0) holds and $\mathcal{F}_1 = C(X)$, $\mathcal{F}_2 = C_f$.

To see it consider an $\mathcal{A}$-measurable $\xi : \Omega \to X$ and let $f(x,\omega) = \xi(\omega)$ for $(x,\omega) \in X \times \Omega$. Then $f^n(x,\omega) = \xi(\omega_n)$ for $(x,\omega) \in X \times \Omega^\infty$, so $\pi_n^f(x,B) = P(\xi \in B) = \pi^f(B)$ for $n \in \mathbb{N}$, $x \in X$, $B \in \mathcal{B}$, and $\int_X F d\pi^f = \int_\Omega F \circ \xi dP$ for $F \in C(X)$. Consequently for every $F \in C(X)$ the function $F - \frac{1}{2}\int_X F d\pi^f$ solves (1), and every $F \in C_f$ solves (2).

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