

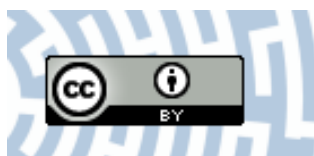


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Author: Tqo cp"I gt"

Citation style: I gt"Tqo cp. (2021). Qp"cnkpcvqp"qh'vy q'hwpevqpcn'gs wcvkpu"qh's wcf tcvk"v(r g.
"Cgs wcvkppgu'O cyj go cvkcg" (2021), doi 3208229.10.22232/243/22: 2;/9



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On alienation of two functional equations of quadratic type

ROMAN GER 

Dedicated to Professor Ludwig Reich on the occasion of his 80th birthday.

Abstract. We deal with an alienation problem for an Euler–Lagrange type functional equation

$$f(\alpha x + \beta y) + f(\alpha x - \beta y) = 2\alpha^2 f(x) + 2\beta^2 f(y)$$

assumed for fixed nonzero real numbers $\alpha, \beta, 1 \neq \alpha^2 \neq \beta^2$, and the classic quadratic functional equation

$$g(x + y) + g(x - y) = 2g(x) + 2g(y).$$

We were inspired by papers of Kim et al. (Abstract and applied analysis, vol. 2013, Hindawi Publishing Corporation, 2013) and Gordji and Khodaei (Abstract and applied analysis, vol. 2009, Hindawi Publishing Corporation, 2009), where the special case $g = \gamma f$ was examined.

Mathematics Subject Classification. 39B52, 39B82.

Keywords. Functional equations, Alienation, Quadratic type equations, Polynomial functions, Székelyhidi's theorem.

Following Rassias [6] numerous authors (S. Abramovich, M. Arunkumar, I. S. Chang, Y. N. Dehghan, G. Eskandani, P. Găvruta, S. Ivelić, K. W. Jun, H. M. Kim, J. Lee, C. Park, J. Pečarić, K. Ravi, J. Son, H. Vaezi, among others) were using the slightly misleading name

Euler–Lagrange equations

referring to the so called Euler–Lagrange (algebraic) identity. Misleading while keeping in mind the celebrated second order partial differential equation of Euler–Lagrange occurring in the calculus of variations. Nevertheless, to prevent possible misunderstanding for the sake of uniformity, we shall keep this terminology in the present paper.

Below we present two examples of Rassias' type Euler–Lagrange equations one may find in the literature:

$$\begin{aligned} f(\alpha x + \beta y) + f(\beta x - \alpha y) &= 2(\alpha^2 + \beta^2)[f(x) + f(y)], \\ f(\alpha x + \beta y) + f(\alpha x - \beta y) &= \frac{\beta(\alpha + \beta)}{2}[f(x + y) + f(x - y)] \\ &\quad + (2\alpha^2 - \alpha\beta - \beta^2)f(x) + (\beta^2 - \alpha\beta)f(y). \end{aligned}$$

In what follows we deal with an alienation problem for another Euler–Lagrange type functional equation

$$f(\alpha x + \beta y) + f(\alpha x - \beta y) = 2\alpha^2 f(x) + 2\beta^2 f(y) \quad (1)$$

and the classic quadratic functional equation

$$g(x + y) + g(x - y) = 2g(x) + 2g(y). \quad (2)$$

We were inspired by papers of Kim et al. [4] and Gordji and Khodaei [2], where the special case $g = \gamma f$ was examined. Among others, in [4, Corollary 5] they proved that if a map f between a normed space $(X, \|\cdot\|)$ and a Banach space $(Y, \|\cdot\|)$ satisfies the functional equation

$$\begin{aligned} f(\alpha x + \beta y) + f(\alpha x - \beta y) + \gamma(f(x + y) + f(x - y) - 2f(x) - 2f(y)) \\ = 2\alpha^2 f(x) + 2\beta^2 f(y) \end{aligned} \quad (3)$$

for all $x, y \in X$, with fixed nonzero real numbers α, β, γ such that $1 \neq \alpha^2 \neq \beta^2$, then f is quadratic provided that $f(0) = 0$ and α is rational.

No word has been said what happens in the case where the coefficient α fails to be rational. The following example shows that, in general, in the case of irrational α the assertion mentioned is no longer valid even in the case where $X = Y = \mathbb{R}$ —the real line.

Example. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a nonzero additive function such that $f(\pi x) = \pi^2 f(x)$, $x \in \mathbb{R}$. (The existence of such functions may be derived from Kuczma's monograph [5, Theorem 4.12.1], for instance; clearly each such function is discontinuous.) Since f is odd (as an additive function) it fails to be quadratic because, otherwise, being even it would vanish identically on X , a contradiction. On the other hand, for every $x, y \in \mathbb{R}$ and each arbitrarily fixed $\beta \in \mathbb{R}$ one has

$$\begin{aligned} f(\pi x + \beta y) + f(\pi x - \beta y) - \beta^2(f(x + y) + f(x - y) - 2f(x) - 2f(y)) \\ = 2f(\pi x) + 2\beta^2 f(y) = 2\pi^2 f(x) + 2\beta^2 f(y), \end{aligned}$$

which proves that f yields a solution of Eq. (3) with $\alpha = \pi$ and $\gamma = -\beta^2$.

It seems worthwhile to remark also that the norm structures in real linear spaces X and Y in Kim–Han–Shim’s result are superfluous.

Observe that Eq. (3) results from (1) and (2) with $g = \gamma f$ by summing them side by side. In what follows we will examine a more general equation

$$f(\alpha x + \beta y) + f(\alpha x - \beta y) + g(x + y) + g(x - y) - 2g(x) - 2g(y) = 2\alpha^2 f(x) + 2\beta^2 f(y), \quad (4)$$

with a quadratic difference of an arbitrary function g instead of that of γf .

Our chief concern will be to answer the question when Eq. (4) forces the functions f and g to satisfy Eqs. (1) and (2), respectively; in other words, whether Eqs. (1) and (2) are alien. The Example above shows that in the case where α is irrational, in general, this is not the case.

Let us start with establishing the following two facts.

Fact 1. Since Eq. (4) is linear the even parts e_f, e_g of f and g , as well as the odd parts o_f, o_g of f and g , are solutions of (4); here

$$e_h(x) := \frac{1}{2}(h(x) + h(-x)), \quad x \in X,$$

$$\text{and } o_h(x) := \frac{1}{2}(h(x) - h(-x)), \quad x \in X,$$

where $h : X \rightarrow Y$ is a function.

Fact 2. Since Eq. (4) gives the equality

$$g(0) = (1 - \alpha^2 - \beta^2) f(0),$$

the functions $f - f(0)$ and $g - g(0)$ satisfy Eq. (4) whenever so do the functions f and g . Therefore, without loss of generality, we may assume that we deal with Eq. (4) jointly with $f(0) = 0$ and $g(0) = 0$.

Lemma. *If an odd function $\varphi : X \rightarrow Y$ satisfies equation*

$$\frac{1}{\gamma^2} [\varphi(x + \gamma y) + \varphi(x - \gamma y)] = \varphi(x + y) + \varphi(x - y) + \frac{2}{\gamma^2} (1 - \gamma^2) \varphi(x) \quad (\gamma)$$

for all $x, y \in X$ where γ is a real number with $0 \neq \gamma^2 \neq 1$, then φ is a polynomial function of at most third degree. More precisely, there exist a symmetric map $A_3 : X^3 \rightarrow Y$ enjoying the property

$$A_3(x, \gamma y, \gamma y) = \gamma^2 A_3(x, y, y), \quad x, y \in X, \quad (A_3)$$

that is additive in each variable, and an additive map $a : X \longrightarrow Y$ such that

$$\varphi(x) = A^3(x) + a(x), x \in X,$$

where $A^3(x) := A_3(x, x, x)$, $x \in X$.

Proof. Since $\gamma^2 \neq 1$ Eq. (γ) may be rewritten in the form

$$\begin{aligned} & \varphi(x) + \frac{\gamma^2}{2(1-\gamma^2)}\varphi(x+y) \\ & + \frac{\gamma^2}{2(1-\gamma^2)}\varphi(x-y) - \frac{1}{2(1-\gamma^2)}\varphi(x+\gamma y) \\ & - \frac{1}{2(1-\gamma^2)}\varphi(x-\gamma y) = 0. \end{aligned}$$

On account of (a modified) Theorem 9.5 from Székelyhidi's monograph [7] we infer that φ is a polynomial function of at most 3-rd degree, i.e.

$$\varphi(x) = A^3(x) + A^2(x) + a(x) + c, \quad x \in X;$$

here $A^k(x) = A_k(x, \dots, x)$, $x \in X$, where $A_k : X^k \longrightarrow Y$ is a symmetric map that is additive in each variable, $k \in \{2, 3\}$, $a : X \longrightarrow Y$ is additive and $c \in Y$ is a constant. Due to the oddness of φ we have $\varphi(0) = 0$ whence $c = 0$ and the summand A^2 disappears. Consequently,

$$\varphi(x) = A^3(x) + a(x), \quad x \in X. \quad (5)$$

Now, applying representation (5) jointly with the well known addition formula

$$A^3(u+v) = A^3(u) + 3A_3(u, u, v) + 3A_3(u, v, v) + A^3(v), \quad u, v \in X,$$

and (γ) , on account of some simple calculations, we deduce that (A_3) holds true for all $x, y \in X$, as claimed.

Note that the equality (A_3) is always satisfied provided that γ is rational. \square

Proposition 1. *If functions f and g satisfying Eq. (4) are odd then both f and g are polynomial functions of at most third degree provided that $\alpha \neq 0 \neq \beta$ and $\alpha^2 \neq \beta^2$. More precisely, there exist a symmetric map $A_3 : X^3 \longrightarrow Y$ enjoying the properties*

$$\begin{aligned} & A_3(\alpha x, \beta y, \beta y) = \beta^2 A_3(x, y, y) \text{ and} \\ & \alpha^2 A_3(x, \beta y, \beta y) = \beta^2 A_3(x, \alpha y, \alpha y), \quad x, y \in X, \end{aligned} \quad (6)$$

that is additive in each variable, and an additive map $a : X \longrightarrow Y$ satisfying the condition $a(\alpha x) = \alpha^2 a(x)$ for all $x \in X$ such that

$$f(x) = A^3(x) + a(x), \quad x \in X, \quad \text{and} \quad g(x) = -\beta^2 f(x), \quad x \in X,$$

where $A^3(x) := A_3(x, x, x)$, $x \in X$.

Conversely, each pair (f, g) of such functions yields a solution to Eq. (4).

Proof. On setting $x = 0$ in (4), we get

$$g(y) = -\beta^2 f(y), \quad y \in X, \quad (*)$$

whence

$$\begin{aligned} f(\alpha x + \beta y) + f(\alpha x - \beta y) - \beta^2 [f(x + y) + f(x - y) - 2f(x) - 2f(y)] \\ = 2\alpha^2 f(x) + 2\beta^2 f(y) \end{aligned}$$

or, equivalently,

$$\begin{aligned} f(\alpha x + \beta y) + f(\alpha x - \beta y) \\ = \beta^2 [f(x + y) + f(x - y)] + 2(\alpha^2 - \beta^2)f(x) \end{aligned} \quad (**)$$

for all $x, y \in X$. In particular, with $y = 0$ Eq. (**) implies that

$$f(\alpha x) = \alpha^2 f(x) \quad \text{for all } x \in X. \quad (***)$$

Since $\alpha \neq 0$ Eq. (**) may be written in the following form:

$$\begin{aligned} f\left(\alpha\left(x + \frac{\beta}{\alpha}y\right)\right) + f\left(\alpha\left(x - \frac{\beta}{\alpha}y\right)\right) \\ = \beta^2 [f(x + y) + f(x - y)] + 2(\alpha^2 - \beta^2)f(x) \end{aligned}$$

whence with the aid of (**), by setting $\gamma := \beta/\alpha$, we infer that

$$f(x + \gamma y) + f(x - \gamma y) = \gamma^2 [f(x + y) + f(x - y)] + 2(1 - \gamma^2)f(x).$$

In view of the inequality $\beta \neq 0$ this states nothing else but (γ) with $\varphi = f$. Therefore, by means of the Lemma, there exist a symmetric map $A_3 : X^3 \longrightarrow Y$ enjoying the property (A_3) , that is additive in each variable, and an additive map $a : X \longrightarrow Y$ such that

$$f(x) = A^3(x) + a(x), \quad x \in X, \quad (5f)$$

where $A^3(x) := A_3(x, x, x)$, $x \in X$. In view of the definition of γ condition (A_3) may equivalently be rewritten as

$$\alpha^2 A_3(x, \beta y, \beta y) = \beta^2 A_3(x, \alpha y, \alpha y), \quad x, y \in X,$$

which gives the second part of (6).

Finally, from (5f) and $(**)$ we have

$$\varphi(x) := A^3(\alpha x) - \alpha^2 A^3(x) = \alpha^2 a(x) - a(\alpha x) =: \psi(x)$$

valid for all $x \in X$. Clearly, for all $x \in X$, we have also

$$8\varphi(x) = \varphi(2x) = \psi(2x) = 2\psi(x)$$

whence

$$4\varphi(x) = \psi(x) = \varphi(x),$$

which states that φ vanishes identically on X , i.e.

$$A^3(\alpha x) = \alpha^2 A^3(x), \quad x \in X, \quad \text{and} \quad a(\alpha x) = \alpha^2 a(x), \quad x \in X. \quad (7)$$

To get the first part of (6) it remains to apply representation (5f) in $(**)$ and to use the addition formula for A^3 jointly with (7).

To finish the proof it remains to perform a mechanical calculation showing that each pair (f, g) of functions described above yields a solution to Eq. (4). \square

Remark 1. Conditions (6) are always satisfied whenever $\alpha = 1$ and β is rational.

Remark 2. In the case where both nonzero coefficients α and β are rational and $\alpha \neq 1$, the first condition in (6) forces the function

$$X^2 \ni (x, y) \longmapsto A_3(x, y) \in Y$$

to vanish. In particular, $A^3 = 0$ and, consequently, the only odd solution of Eq. (4) is just the zero function because the additive summand a in representation (5f) has to satisfy (7) and being rationally homogeneous it has to vanish as well.

Proposition 2. *If functions f and g satisfying Eq. (4) jointly with $f(0) = g(0) = 0$ are even then both f and g are quadratic provided that $1 \neq \alpha^2 \neq \beta^2 \neq 0$ and α is a rational number.*

Proof. On setting $y = 0$ (resp. $x = 0$) in (4), we get

$$f(\alpha x) = \alpha^2 f(x), \quad x \in X, \quad (\text{resp. } f(\beta y) = \beta^2 f(y), \quad y \in X). \quad (8)$$

Put $F := g + \alpha^2 f$; then

$$\begin{aligned} F(x) - \frac{1}{2}f(\alpha x + \beta y) - \frac{1}{2}f(\alpha x - \beta y) - \frac{1}{2}g(x + y) \\ - \frac{1}{2}g(x - y) + h(y) = 0, \quad x, y \in X, \end{aligned}$$

with $h := g + \beta^2 f$. On account of (a modified) Theorem 9.5 from Székelyhidi's monograph [7] we infer that F is a polynomial function of at most 4-th degree, i.e.

$$g(x) + \alpha^2 f(x) = F(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x), \quad x \in X; \quad (9)$$

here $A^k(x) = A_k(x, \dots, x)$, $x \in X$, where $A_k : X^k \rightarrow Y$ is a symmetric map that is additive in each variable, $k \in \{1, 2, 3, 4\}$. The evenness of f and g imply the evenness of F whence $A^3 = A^1 = 0$. Consequently, in view of (9), we get

$$g(x) = A^4(x) + A^2(x) - \alpha^2 f(x), \quad x \in X.$$

With the aid of the well known addition formulas for A^4 and A^2 :

$$\begin{aligned} A^4(x+y) &= A^4(x) + 4A_4(x, x, x, y) \\ &\quad + 6A_4(x, x, y, y) + 4A_4(x, y, y, y) + A^4(y), \quad x, y \in X, \end{aligned}$$

and

$$A^2(x+y) = A^2(x) + 2A_2(x, y) + A^2(y), \quad x, y \in X,$$

a simple calculation shows that the equality

$$\begin{aligned} &g(x+y) + g(x-y) - 2g(x) - 2g(y) \\ &= 12A_4(x, x, y, y) - \alpha^2 [f(x+y) + f(x-y) - 2f(x) - 2f(y)] \end{aligned}$$

is valid for all $x, y \in X$.

Now, Eq. (4) assumes the form

$$\begin{aligned} &f(\alpha x + \beta y) + f(\alpha x - \beta y) - \alpha^2 [f(x+y) + f(x-y) - 2f(x) - 2f(y)] \\ &\quad + 12A_4(x, x, y, y) \\ &= 2\alpha^2 f(x) + 2\beta^2 f(y), \quad x, y \in X. \end{aligned}$$

Replacing here x by αx and y by αy and taking the first equality in (8) into account along with the fact that α is a nonzero rational number we obtain

$$\begin{aligned} &f(\alpha x + \beta y) + f(\alpha x - \beta y) \\ &\quad - \alpha^2 [f(x+y) + f(x-y) - 2f(x) - 2f(y)] + 12\alpha^2 A_4(x, x, y, y) \\ &= 2\alpha^2 f(x) + 2\beta^2 f(y), \quad x, y \in X. \end{aligned}$$

The latter two equations imply that the equality

$$(\alpha^2 - 1)A_4(x, x, y, y) = 0$$

is satisfied for all $x, y \in X$. Since, by assumption, $\alpha^2 \neq 1$ we infer that $A_4(x, x, y, y) = 0$ for all $x, y \in X$. Consequently, one has

$$\begin{aligned} & f(\alpha x + \beta y) + f(\alpha x - \beta y) \\ & \quad - \alpha^2 [f(x + y) + f(x - y) - 2f(x) - 2f(y)] \\ & = 2\alpha^2 f(x) + 2\beta^2 f(y), \quad x, y \in X. \end{aligned} \quad (10)$$

Putting here βx and αy in place of x and y , respectively, we obtain the equality

$$\begin{aligned} & f(\alpha\beta(x + y)) + f(\alpha\beta(x - y)) - \alpha^2 [f(\beta x + \alpha y) \\ & \quad + [f(\beta x - \alpha y) - 2f(\beta x) - 2f(\alpha y)]] \\ & = 2\alpha^2 f(\beta x) + 2\beta^2 f(\alpha y) \end{aligned}$$

which, on account of (8) and the fact that $\alpha \neq 0$, gives now

$$\begin{aligned} & \beta^2 f(x + y) + \beta^2 f(x - y) \\ & \quad - [f(\beta x + \alpha y) + f(\beta x - \alpha y) - 2f(\beta x) - 2f(\alpha y)] \\ & = 2\beta^2 f(x) + 2\beta^2 f(y), \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \beta^2 [f(x + y) + f(x - y) \\ & \quad - 2f(x) - 2f(y)] = f(\beta x + \alpha y) + f(\beta x - \alpha y) - 2f(\beta x) - 2f(\alpha y). \end{aligned}$$

Interchanging here the roles of x and y and applying (8) jointly with the evenness of f we infer that

$$\begin{aligned} & \beta^2 [f(x + y) + f(x - y) - 2f(x) - 2f(y)] = f(\alpha x + \beta y) + f(\alpha x - \beta y) \\ & \quad - 2\beta^2 f(y) - 2\alpha^2 f(x). \end{aligned}$$

Therefore, in view of (10), we conclude that the equality

$$(\alpha^2 - \beta^2) [f(x + y) + f(x - y) - 2f(x) - 2f(y)] = 0$$

holds true for all $x, y \in X$. By assumption $\alpha^2 \neq \beta^2$ which forces f to be quadratic. Thus the proof has been completed because the quadraticity of g results now easily from (4), (10) and the quadraticity of f . \square

Theorem 1. *Let X, Y be two real linear spaces and let α, β be two nonzero real numbers such that*

$$1 \neq \alpha^2 \neq \beta^2.$$

If α is rational then equations

$$f(\alpha x + \beta y) + f(\alpha x - \beta y) = 2\alpha^2 f(x) + 2\beta^2 f(y) \quad (1)$$

and

$$g(x + y) + g(x - y) = 2g(x) + 2g(y) \quad (2)$$

are alien in the class of all mappings $f, g : X \longrightarrow Y$ vanishing at zero, i.e. equation

$$\begin{aligned} f(\alpha x + \beta y) + f(\alpha x - \beta y) + g(x + y) + g(x - y) - 2g(x) - 2g(y) & \quad (4) \\ = 2\alpha^2 f(x) + 2\beta^2 f(y) \end{aligned}$$

is satisfied if and only if both Eqs. (1) and (2) are fulfilled provided that $f(0) = g(0) = 0$.

Moreover, if that is the case, then f is quadratic.

Proof. Let a pair (f, g) of mappings from X into Y be a solution of Eq. (4). Since the pairs (o_f, o_g) , (e_f, e_g) yield also solutions of (4) (cf. Fact 1), Proposition 1 guarantees the existence of a symmetric map $A_3 : X^3 \longrightarrow Y$ enjoying the properties (6), that is additive in each variable, and an additive map $a : X \longrightarrow Y$ satisfying the condition $a(\alpha x) = \alpha^2 a(x)$ for all $x \in X$ such that

$$\begin{aligned} o_f(x) &= A^3(x) + a(x), \quad x \in X, \quad \text{and} \quad o_g(x) \\ &= -\beta^2 o_f(x), \quad x \in X, \end{aligned}$$

where $A^3(x) = A_3(x, x, x)$, $x \in X$. Moreover, conditions (7) are satisfied which jointly with the rationality of the coefficient $\alpha \notin \{0, 1\}$ imply that $A^3 = a = 0$. Consequently, $o_f = o_g = 0$, i.e. both f and g are even.

On the other hand, due to the rationality of α , Proposition 2 states then that both f and g are quadratic. In particular, g satisfies Eq. (2) which jointly with (4) implies the validity of (1). Therefore we are faced with the alienation of Eqs. (1) and (2), as claimed.

Since the converse implication is trivial the proof has been completed.

As an easy consequence we get now the following □

Theorem 2. *Let X, Y be two real linear spaces and let α, β be two nonzero real numbers such that*

$$1 \neq \alpha^2 \neq \beta^2.$$

If α is rational then Eqs. (1) and (2) are alien modulo a constant in the class of all mappings $f, g : X \longrightarrow Y$. More precisely, Eq. (4) is satisfied if and only if there exists a constant $c \in Y$ such that

$$f(\alpha x + \beta y) + f(\alpha x - \beta y) = 2\alpha^2 f(x) + 2\beta^2 f(y) + c \quad (1')$$

and

$$g(x+y) + g(x-y) = 2g(x) + 2g(y) - c. \quad (2')$$

Moreover, if that is the case and $g(0) \neq 0$, then $\alpha^2 + \beta^2 \neq 1$ and $f = F + \frac{c}{2(1-\alpha^2+\beta^2)}$ whereas $g = G + \frac{1}{2}c$, with quadratic mappings F and G .

Proof. Let a pair (f, g) of mappings from X into Y be a solution of Eq. (4). An appeal to Fact 2 shows that so is the pair (F, G) of mappings $F := f - f(0)$ and $G := g - g(0)$. Since, obviously, $F(0) = G(0) = 0$, Theorem 1 implies that

$$F(\alpha x + \beta y) + F(\alpha x - \beta y) = 2\alpha^2 F(x) + 2\beta^2 F(y)$$

and

$$G(x+y) + G(x-y) = 2G(x) + 2G(y).$$

In terms of f and g the latter system states that we are faced with relations (1') and (2') with $c := 2(1-\alpha^2-\beta^2)f(0)$ because of the equality $g(0) = (1-\alpha^2-\beta^2)f(0)$ resulting from Eq. (4) on setting $x = y = 0$.

It is a straightforward matter to verify the remaining statements. \square

Corollary. Fix a real constant γ and put $g = \gamma f$ in Eq. (4) to get an Euler–Lagrange type equation

$$\begin{aligned} f(\alpha x + \beta y) & \hspace{15em} \text{(KHS)} \\ & + f(\alpha x - \beta y) + \gamma[f(x+y) + f(x-y) - 2f(x) - 2f(y)] \\ & = 2\alpha^2 f(x) + 2\beta^2 f(y), \end{aligned}$$

which was examined in paper [4] by Chang Il Kim, Giljun Han and Seong-A. Shim. The basic results on solutions of (HKS) established in [4] become special cases of Theorem 1 from the present paper.

Remark 3. Equation (HKS) admits nontrivial solutions that are not quadratic (see Proposition 1 and the Example above). They were omitted in [4] because the authors were dealing mainly with rational coefficients α in (HKS).

Remark 4. I presented the results of the present paper during the 20th Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities that was held in Hajdúszoboszló (Hungary) from January 29 till February 1, 2020. At that time my proofs were entirely different. In particular, Theorem 4 from Kim–Han–Shim paper [4] and, indirectly,

Theorem 2.1 from Jun–Kim–Chang’s paper [3] were used as proof tools. During the discussion after my talk, a Hungarian mathematician Mihály Bessenyei (see [1]) asked me the question whether such approach was unavoidable since in that case my generalization of results from [4] is just formal only. During the meeting I was unable to answer Bessenyei’s question but it mobilized me to work on that problem. Finally, I succeeded to get a positive answer using a celebrated theorem of Székelyhidi from [7].

Acknowledgements

I wish to thank Mihály Bessenyei at this place for his valuable inspiration.

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Received: October 16, 2020

Revised: May 13, 2021

Accepted: May 19, 2021