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On Some Functional Equations Related to Alpha Migrative t-conorms

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Abstract

In this contribution, we analyse in details the recently introduced definition of migrative t-conorms [see *Fuzzy implications: alpha migrativity and generalised laws of importation*, M. Baczyński, B. Jayaram, R. Mesiar, 2020]. We also focus on some general functional equations, which might be obtained from such a notion. We concentrate on some particular well-known families of fuzzy implications and show solutions of those equations among this kind of fuzzy implication functions.

Keywords: Fuzzy connectives, T-conorm, Fuzzy implication, Migrativity.

1 Introduction

A notion of α-migrativity appeared in 2007, where authors dealt with the problem of a convex combination of t-norms [8]. It is useful from an application point of view since it shows when "proportional" change of one of the coordinates is equivalent of such a change for another argument. Therefore it is important which aggregation functions can ensure this property.
Below we present the original definition of α-migrativity, which is a starting point for further investigations.

Definition 1.1 ([6, Definition 1]). Let $\alpha \in (0,1)$. A binary operator $T\colon [0,1]^2 \to [0,1]$ is said to be α-migrative if it satisfies

$$T(\alpha x, y) = T(x, \alpha y), \quad x \in [0,1].$$

A function T is called migrative if it is α-migrative for any $\alpha \in [0,1]$.

Although a convex combination of two t-norms is not a t-norm, a notion of migrativity has been widely investigated in a context of different fuzzy connectives ([13]) and related concepts like bimigrativity (see [11]), or migrativity over t-norms [12]. One of the earliest papers provides the following fact.

Theorem 1.2 ([5, Corollary 2]). *There are no migrative t-conorms, uninorms or nullnorms (in the sense of Definition 1.1).*

The above sentence is, of course, correct but only under the consideration of Definition 1.1. However, other authors investigated migrativity for different fuzzy connectives – with another definition. Namely, in [3] fuzzy implications were examined. That particular definition of α-migrativity is associated with the graphical interpretation of it. Moreover, the authors explored functional equations connected with such migrativity. The above-mentioned article was an inspiration for us to examine t-conorms, especially when in [3] authors formulated the desired definition.

The paper is organized as follows. Section 2 contains some important definitions and theorems, while the main results are collected in Section 3.

2 Preliminaries

To make this work more self-contained, we placed some important definitions here.

Definition 2.1 ([9]). Let $n \in \mathbb{N}$. An aggregation function in $[0,1]^n$ is a function $A^{(n)}\colon [0,1]^n \to [0,1]$ which is nondecreasing (in each variable) and it satisfies $A^{(n)}(0,\ldots,0) = 0$ and $A^{(n)}(1,\ldots,1) = 1$.

Definition 2.2 ([4, Definition 1.9]). An aggregation function $A^{(n)}\colon [0,1]^n \to [0,1]$ has disjunctive behavior (or is disjunctive) if for every $x = (x_1,\ldots,x_n)$ it is bounded by

$$A(x) \geq \max(x) = \max(x_1,\ldots,x_n).$$

Definition 2.3 ([7, 10]). A function $T\colon [0,1]^2 \to [0,1]$ is called a **triangular norm** (**t-norm** in short), if it is associative, commutative and non-decreasing operator with a neutral element 1.

Definition 2.4 ([10]). A function $S\colon [0,1]^2 \to [0,1]$ is called a **triangular conorm** (**t-conorm** in short), if it is associative, commutative and non-decreasing operator with a neutral element 0.

Definition 2.5 ([15]). A function $U\colon [0,1]^2 \to [0,1]$ is called a **uninorm**, if it is an associative, commutative and non-decreasing operator with a neutral element $e \in [0,1]$, i.e., such that $U(x,e) = x$, $x \in [0,1]$. A uninorm U such that $U(0,1) = 0$ is called **conjunctive** and if $U(0,1) = 1$, then it is called **disjunctive**.

Definition 2.6 ([2, 10]). A non-increasing function $N\colon [0,1] \to [0,1]$ is called a **fuzzy negation**, if $N(0) = 1$, $N(1) = 0$. Moreover, a fuzzy negation N is called **strict** if it is strictly decreasing and continuous, and **strong** if it is an involution, i.e., $N(N(x)) = x$, for all $x \in [0,1]$.

Theorem 2.7 ([7, Proposition 1.9]). *Let T be a t-norm and N be a strict negation. Then the function $S\colon [0,1]^2 \to [0,1]$ given by*

$$S(x,y) = N^{-1}(T(N(x),N(y))), \quad x,y \in [0,1], \qquad (1)$$

is a t-conorm. We say that S is an N-dual t-conorm to the t-norm T.

Definition 2.8 ([2, 7]). A function $I\colon [0,1]^2 \to [0,1]$ is called a **fuzzy implication**, if it is non-increasing with respect to the first variable, non-decreasing with respect to the second variable and if $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$. The family of fuzzy implications will be denoted by $\mathcal{FI}$.

Definition 2.9 ([2]). We say that a fuzzy implication I satisfies

(i) the **identity principle**, if

$$I(x,x) = 1, \quad x \in [0,1], \qquad \text{(IP)}$$

(ii) the **left neutrality property**, if

$$I(1,y) = y, \quad y \in [0,1]. \qquad \text{(NP)}$$

Definition 2.10 ([2, Definition 2.4.1]). A function $I\colon [0,1]^2 \to [0,1]$ is called an (S,N)-implication, if there exist a t-conorm S and a fuzzy negation N such that

$$I(x,y) = S(N(x),y), \quad x,y \in [0,1].$$

If I is generated from a t-conorm S and a fuzzy negation N, then it will be denoted by $I_{S,N}$.

Definition 2.11 ([2, Definition 5.3.1]). A function $I\colon [0,1] \to [0,1]$ is called a (U,N)-operation, if there exist a uninorm U and a fuzzy negation N such that

$$I(x,y) = U(N(x),y), \quad x,y \in [0,1].$$

A (U,N)-operator $I_{U,N} \in \mathcal{FI}$ if and only if U is a disjunctive uninorm (see [2, Theorem 5.3.3]).

3 Main results

Here, we recall some concepts presented in [3]. The definition of α-migrative t-conorm is based on some particular graphical interpretation of this property (see also Fig. 1).

Definition 3.1 ([3, Definition 4.1]). Let $\alpha \in (0,1)$ be fixed. A t-conorm S is said to be α-migrative, if it satisfies

$$S(1-\alpha+\alpha x, y) = S(x, 1-\alpha+\alpha y), \qquad (2)$$

for all $x,y \in [0,1]$. If S is α-migrative for every $\alpha \in (0,1)$, then S is said to be migrative.

Of course, Eq. (2) is always satisfied when $\alpha = 0$ or $\alpha = 1$. Indeed, for $\alpha = 0$ we have

$$S(1,y) = 1 = S(x,1)$$

and for $\alpha = 1$ we have

$$S(x,y) = S(x,y).$$

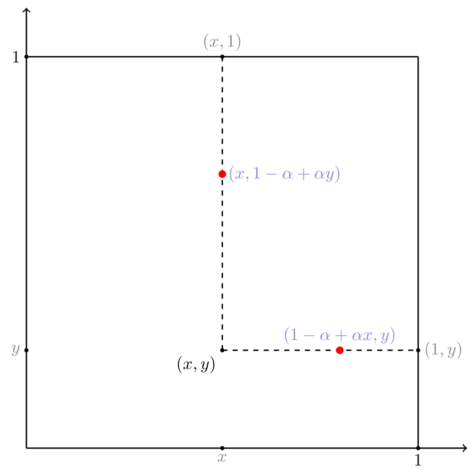

Figure 1: The graphical interpretation of α-migrative t-conorm (see Definition 3.1)

First, let us formulate simple facts.

Proposition 3.2. *Let T be an α-migrative t-norm for some $\alpha \in (0,1)$ (in the sense of Definition 1.1). If S is the $N_{\mathbf{C}}$-dual t-conorm to T, where $N_C(x) = 1-x$, for all $x \in [0,1]$, then S is α-migrative (in the sense of Definition 3.1).*

Proof. Let T be an α-migrative t-norm for some $\alpha \in (0,1)$ and S be te $N_{\mathbf{C}}$-dual t-conorm to T. Then we have

$$\begin{aligned} S(1-\alpha+\alpha x,y) &= 1-T(1-(1-\alpha+\alpha x),1-y) \\ &= 1-T(\alpha(1-x),1-y) \\ &= 1-T(1-x,\alpha(1-y)) \\ &= 1-T(1-x,1-(1-\alpha+\alpha y)) \\ &= S(x,1-\alpha+\alpha y), \end{aligned}$$

for any $x,y \in [0,1]$. □

We can also show the reverse of Proposition 3.2; the $N_{\mathbf{C}}$-dual t-norm to α-migrative t-conorm is also α-migrative. Based on [5, Theorem 7] we know that the only migrative t-norm is the product t-norm. Hence the following fact is easy to obtain.

Corollary 3.3. *S_P is the only migrative t-conorm.*

Furthermore, inspired by investigations done in [3], we can consider the following functional equation, which is nothing else but the other form of the formula from Definition 3.1:

$$S(I_{\mathbf{RC}}(\alpha,x),y) = S(x,I_{\mathbf{RC}}(\alpha,y)), \quad \alpha,x,y \in [0,1], \tag{3}$$

where $I_{\mathbf{RC}}$ is the Reichenbach implication given by $I_{\mathbf{RC}}(x,y) = 1-x+xy$. Hence, as the Pexider type of the above equation, we can analyse the following functional equation

$$S_1(I_1(\alpha,x),y) = S_2(x,I_2(\alpha,y)), \tag{4}$$

satisfied for all $\alpha,x,y \in [0,1]$, where S_1,S_2 are t-conorms (or any generalisations of the classical disjunction) and I_1,I_2 are fuzzy implications (or any generalisation of the classical implication).

Let us start with some necessary conditions for functions S_1,S_2,I_1,I_2 to satisfy (4).

Proposition 3.4. *Let S_1,S_2 be aggregation functions, $I_1,I_2 \in \mathcal{FI}$. If (S_1,S_2,I_1,I_2) satisfies* (4)*, then*

(i) 1 is an annihilator of S_1, S_2,

(ii) if I_1,I_2 satisfies (NP)*, then $S_1 = S_2$.*

Proof. (i) : Let us take $x=1$ and $\alpha=0$ in (4). Then we have

$$\begin{aligned} S_1(I_1(0,1),y) = S_2(1,I_2(0,y)) &\iff S_1(1,y) = S_2(1,1) \\ &\iff S_1(1,y) = 1, \end{aligned}$$

for all $y \in [0,1]$, so 1 is an annihilator of S_1.

Now, let us put $y=1$ and $\alpha=0$ in (4). Then we have

$$\begin{aligned} S_1(I_1(0,x),1) = S_2(x,I_2(0,1)) &\iff S_1(1,1) = S_2(x,1) \\ &\iff 1 = S_2(x,1), \end{aligned}$$

for all $x \in [0,1]$, so 1 is an annihilator of S_2.

(ii) : Let us take $\alpha=1$ in (4). Then we obtain

$$S_1(I_1(1,x),y) = S_2(x,I_2(1,y)) \iff S_1(x,y) = S_2(x,y)$$

for all $x,y \in [0,1]$. □

Next results can be obtained in a natural way.

Proposition 3.5. *Let N be a fuzzy negation and U be a uninorm. Then the quadruple $(U,U,I_{U,N},I_{U,N})$ satisfies* (4).

Proposition 3.6. *Let $I_1,I_2 \in \mathcal{FI}$ and S_1 (S_2) be binary functions. Next, let us assume that the quadruple (S_1,S_2,I_1,I_2) satisfies* (4)*. If S_1 (S_2, respectively) is a t-conorm, then I_1 (I_2, respectively) is an (S_2,N_{I_2})-implication ((S_1,N_{I_1})-implication, respectively).*

After these results, it is clear that we should focus on (U,N)-implications. Note that we can expand Definition 2.11 and talk about (A,N)-operators, where A is just an aggregation function. Also, the following result is obvious.

Proposition 3.7 ([14, Theorem 33])**.** *If A is a disjunctive aggregation function and N is a fuzzy negation, then the operator $I\colon [0,1]^2 \to [0,1]$ given by the formula $I(x,y) = A(N(x),y)$ is a fuzzy implication.*

Moreover, from the above investigations, it seems that usually in Eq. (4) we assume that $I_1 = I_2$. This does not always have to be true. Let us analyse the following functions.

Example 3.8. Let us take the classical fuzzy negation $N_{\mathbf{C}}$ and the following aggregation operator

$$A(x,y) = \begin{cases} 0, & x=y=0, \\ 1, & \text{otherwise.} \end{cases}$$

Then

$$I_{A,N_{\mathbf{C}}}(x,y) = \begin{cases} 0, & x=1, y=0, \\ 1, & \text{otherwise.} \end{cases}$$

Also, let us consider the drastic (S,N)-implication given by the formula

$$I_{\mathbf{D}}(x,y) = \begin{cases} 1, & x=0, \\ y, & \text{otherwise.} \end{cases}$$

Then we have $A(I_{(A,N)}(\alpha,x),y) = A(x,I_{\mathbf{D}}(x,y))$, so the quadruple $(A,A,I_{A,N_{\mathbf{C}}},I_{\mathbf{D}})$ satisfies (4).

Also, it is not true that always (according to the above notation) $I_1 = I_{S_1,N}$ for some N.

Example 3.9. Again, let us take $I_{\mathbf{D}}$ and any t-conorm S. Then it can be easily shown that the quadruple $(S,S,I_{\mathbf{D}},I_{\mathbf{D}})$ satisfies (4).

Let us go back to (A,N)-operators. If we consider that $I = I_{A,N}$ in (4), then we obtain the following functional equation:

$$S_1(A_1(N(\alpha),x),y) = S_2(x,A_2(N(\alpha),y)),$$

satisfied for $\alpha,x,y \in [0,1]$. With the assumptions of the commutativity of A_1,A_2 and the surjectivity of N, it leads us to the following functional equation of generalized associativity:

$$S_1(A_1(x,z),y) = S_2(x,A_2(z,y)),\ x,y,z \in [0,1]. \quad \text{(GA)}$$

This equation has been already investigated by Aczěl et al. [1, Theorem 1]. However, the authors required there functions S_1,S_2,A_1,A_2 with properties, which were ensured by the existence of a quasigroup. Therefore, we should add properties that can substitute those assumptions for fuzzy connectives and still give a valid result. To obtain it, we restrict our considerations to some particular aggregation functions.

Definition 3.10. We say that an aggregation function $A\colon [0,1]^2 \to [0,1]$ is "quasi-multiplicative" generated if there exist continuous bijections $f,g,h\colon [0,1] \to [0,1]$ such that

$$A(x,y) = h(f(x)\cdot g(y)), \quad x,y \in [0,1], \qquad \text{(QM)}$$

where f,g,h are all increasing (or all decreasing) with

$$h^{-1}(0) = f(0)g(0) \text{ and } h^{-1}(1) = f(1)g(1).$$

Note that strict t-norms and t-conorms are subclasses of "quasi-multiplicative" generated functions. Thus, we have the following interesting result for the introduced family of aggregation functions.

Theorem 3.11. *Let S_1,S_2,A_1,A_2 be "quasi-multiplicative" generated aggregation functions. Then the following statements are equivalent:*

(i) The quadruple (S_1,S_2,A_1,A_2) satisfies (GA).

(ii)
$$A_1(x,y) = \varphi^{-1}(G(\xi(x),\lambda(y))),$$
$$S_1(x,y) = G(\varphi(x),\psi(y)),$$
$$A_2(x,y) = \rho^{-1}(G(\lambda(x),\psi(y))),$$
$$S_2(x,y) = G(\xi(x),\rho(y)),$$

for $x,y \in [0,1]$, where $\varphi,\psi,\xi,\rho,\lambda\colon [0,1] \to [0,1]$ are continuous bijections and $G\colon [0,1]^2 \to [0,1]$ is an associative function.

Let us finish our manuscript with the following interesting examples.

Example 3.12. (i) Let

$$\varphi(x) = x^2,$$
$$\psi(x) = \xi(x) = \lambda(x) = x,$$
$$\rho(x) = x^3,$$

for $x \in [0,1]$ and

$$G(x,y) = x+y-xy, \quad x,y \in [0,1].$$

Then

$$S_1(x,y) = x^2+y-x^2y,$$
$$S_2(x,y) = x+y^3-xy^3,$$
$$A_1(x,y) = \sqrt{x+y-xy},$$
$$A_2(x,y) = \sqrt[3]{x+y-xy},$$

for $x,y \in [0,1]$. From our main new result (Theorem 3.11) we have that the quadruple (S_1,A_1,S_2,A_2) satisfies (GA) (in fact, this observation can be easily checked). Moreover, the quadruple $(S_1,S_2,I_{A_1,N_C},I_{A_2,N_C})$ satisfies (4).

(ii) Note, that if G is, for instance, the product t-norm, then with the above $\varphi,\psi,\xi,\rho,\lambda$, the Eq. (GA) is satisfied by the quadruple (S_1,A_1,S_2,A_2), where

$$S_1(x,y) = x^2y,$$
$$S_2(x,y) = xy^3,$$
$$A_1(x,y) = \sqrt{xy},$$
$$A_2(x,y) = \sqrt[3]{xy^3},$$

for $x,y \in [0,1]$.

4 Conclusions

In this paper, we have discussed the property of migrativity of t-conorms. Also, we have shown how it is connected with the notion of generalized associativity. Moreover, we have given some solutions to such a functional equation.

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