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UNCERTAINTY IN THE CONJUNCTIVE APPROACH TO FUZZY INFERENCE

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Fuzzy inference using the conjunctive approach is very popular in many practical applications. It is intuitive for engineers, simple to understand, and characterized by the lowest computational complexity. However, it leads to incorrect results in the cases when the relationship between a fact and a premise is undefined. This article analyses the problem thoroughly and provides several possible solutions. The drawbacks of uncertainty in the conjunctive approach are presented using fuzzy inference based on a fuzzy truth value, first introduced by Baldwin (1979c). The theory of inference is completed with a new truth function named 0-undefined for two-valued logic, which is further generalized into fuzzy logic as α-undefined. Eventually, the proposed modifications allow altering existing implementations of conjunctive fuzzy systems to interpret the undefined state, giving adequate results.

Keywords: fuzzy inference, conjunctive approach, fuzzy truth value.

1. Introduction

Fuzzy inference based on the conjunctive approach, where a conjunction operator replaces an implication operator, is very popular in many practical applications. Starting from Mamdani and Asilan (1975), over the years many researchers have described the model in books and articles. Since it is not possible to indicate all related literature due to the immense popularity of the subject matter, we only limit ourselves to representative examples (Zimmermann, 1985; Klir *et al*., 1997; Czogała and Łęski, 2000; Rutkowski, 2008; Azzini *et al*., 2008; Czabanski *et al*., 2017; Izquierdo and Izquierdo, 2018; Grzegorzewski *et al*., 2020; Piegat and Dobryakova, 2020).

As an example, let us assume the following inference process based on generalized modus ponendo ponens (Zadeh, 1973):

$$\begin{array}{ll} \text{FACT :} & X \text{ is } A' \\ \text{RULE :} & \text{if } X \text{ is } A \text{ then } Y \text{ is } B \\ \hline \text{CONCLUSION :} & Y \text{ is } B' \end{array} \tag{1}$$

The membership function $\mu_{B'}$ of the inference result B', based on Zadeh's definition (Zadeh, 1975) in conjunctive form, can be obtained as follows:

$$\underset{y\in Y}{\forall} \quad \mu_{B'}(y) = \sup_{x\in X} \left[\mu_{A'}(x) \star_T \mu_A(x) \star_T \mu_B(y)\right], \tag{2}$$

where $\mu_{A'}$, μ_A and μ_B represent membership functions of a fact, a premise and a conclusion, respectively. The values of membership functions are joined by any T-norm (like minimum or product), denoted by $\star_T$.

Assuming the fact that A' is described by a singleton, Eqn. (2) will take the following simplified form:

$$\underset{y\in Y}{\forall} \quad \mu_{B'}(y) = \mu_A(x_i) \star_T \mu_B(y), \tag{3}$$

where x_i represents the unique value in the X domain, for which $\mu_{A'}(x) > 0$, and is equal to 1.

The first successful implementation using this idea was a pioneer solution presented by Mamdani and Assilan (1975). The approach is simple and intuitive, especially for engineers. However, the idea has some flaws, which can be observed particularly when the fact is characterized by higher uncertainty, and its description is too general.

To describe the problem, consider sample situations presented in Fig. 1. The graphs show the most popular cases of relationships between a fact ($\mu_{A'}$ membership function) and a premise (μ_A membership function) in terms of the uncertainty definition.

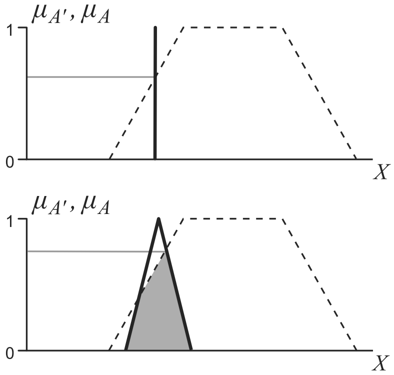

Fig. 1. Most popular relationship between facts and premises in fuzzy systems. Facts are drawn with thick solid lines ($\mu_{A'}$) and premises with a dashed line (μ_A).

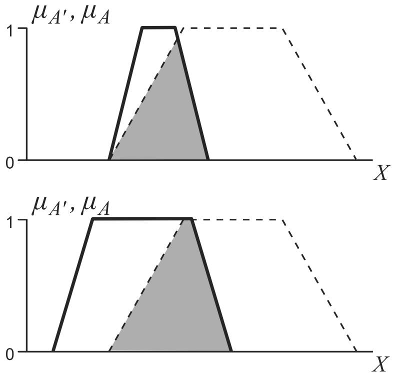

Fig. 2. Relationship between facts and premises for higher levels of uncertainty. Facts are drawn with thick solid lines ($\mu_{A'}$) and premises with a dashed line (μ_A). Intersections are marked as a gray area.

The top graph presents the situation for which a singleton represents the fact. The bottom graph shows the fact fuzzified by a triangular membership function. The type of functions used in these examples does not matter. The goal is to focus on the uncertainty of the fact (the level of uncertainty) and its relationship to the premise.

Both situations are obvious. The conjunctive approach will work perfectly fine because the fact's uncertainty is relatively small to the premise description. Such a situation usually occurs when a given non-fuzzy input must be fuzzified to be applied to a fuzzy reasoning system. Therefore, in short, according to a defined uncertainty, the appropriate fuzzifying function is applied.

The rule activation level in such situations is trivial. In the most simple solutions, like the one presented by Mamdani and Assilan, the minimum function is used as T-norm to compute the compatibility between the fact and the premise. Generally, the higher the maximum value of this intersection, the better. In both examples from Fig. 1 the levels of compatibility between facts and premises, obtained by an intersection using minimum T-norm, are depicted by a gray horizontal line.

Unfortunately, such an approach becomes problematic when the uncertainty of facts goes higher, and their relationship to premises is no longer obvious. Let us consider other situations, presented in Fig. 2.

Both graphs show completely different relations between facts and premises in comparison with the previous examples. The top graph contains a fact characterized by a higher level of uncertainty. However, in this situation, it could be acceptable to compute the compatibility with the premise by calculating the intersection of both membership functions. The fact is just more uncertain, giving a larger range of possibilities for higher compatibility with a premise.

The bottom graph, however, illustrates the disadvantages of such thinking. For larger uncertainty applied to the fact, which can be involved with different reasons in real-life scenarios, the intersection used as a base mechanism in the conjunctive approach gives inadequate results. The situation presented at the bottom of Fig. 2 clearly shows the undefined state. Not true, not false, but undefined, because for different possible values of such a broadly defined fact all possible levels of the premise membership function occur. Therefore, we can no longer assume the premise is true but undefined. In such a case the generalized modus ponendo ponens (1) should generate an undefined conclusion; however, the conjunctive approach will imply that the conclusion is true, which is obviously incorrect.

Although broadly defined facts are rather rarely used on purpose, this does not mean that they cannot occur. Many approaches use automatically computed parameters allowing one to define fuzzification of premises or facts based on input features. Another example could be using an output of one fuzzy system as an input of another one. In such situations, the defined problem is possible to occur.

The article focuses on two main intertwining ideas. The first is the conjunctive approach using the inference model proposed by Baldwin (1979c). This approach has never been analyzed in the literature and it is the first major contribution to fuzzy inference theory. Through this analysis, the problems of the conjunctive approach can be seen, transparently shown with the help of truth functions, naturally occurring in Baldwin's approach. The second important contribution is to propose modifications in the inference models so that, for the situations shown in the example of Fig. 2, they produce results adequate to the level of uncertainty.

2. Related work

Over the years, there have appeared a significant number of articles concerning differences between fuzzy inference mechanisms using implicative and conjunctive approaches. Except for the already mentioned books (i.e., Zimmermann, 1985; Klir *et al.*, 1997; Czogała and Łęski, 2000; Rutkowski, 2008), where the solutions are described in a broader context of fuzzy and intelligent systems, it is worth to point out the articles particularly focusing on differences between inference methods.

One of the first such studies reported was the paper by Mizumoto and Zimmermann (1982), where the authors compared solutions proposed by Zadeh, Mamdani, and Mizumoto for generalized modus ponens and generalized modus tolens (Zadeh, 1973).

Yagger (1996) described the basic characteristics of Mamdani's and logical approaches. Uncertainty in the presented context is not analyzed in this research.

Ughetto *et al.* (1999) as well as Dubois and Prade (1996) described multiple approaches, implicative as well as Mamdani's, focusing on differences in the obtained fuzzy result and possible applications, not the uncertainty itself.

Cordon *et al.* (1997) analyzed and compared the behavior of a large number of fuzzy operators for precisely defined problems. This research also focused on the results and did not consider uncertainty.

Czogała and Kowalczyk (1996) as well as Czogała and Łęski (2001) investigated selected methods for engineering and analyzed an equivalence of approximate reasoning results for the implicative and conjunctive interpretation of if-then rules. The authors considered the equivalence after defuzzification. However, the problem of uncertainty in the simplified environment was not analyzed.

In the context of method comparison it is worth mentioning the research by Kudłacik and Łęski (2021). The authors focus on thorough analysis of Baldwin's and Zadeh's approaches to fuzzy inference, proving the equivalence of both methods in terms of obtained results. Advantages and disadvantages of both mechanisms are shown considering a practical implementation. The research does not address the problem of uncertainty in the context defined in this article.

Advantages of Baldwins's method can be also found in the works of Kudłacik (2010; 2012; 2013). These studies do not analyze the problem of uncertainty in the conjunctive approach, either, and focus on the performance of the inference process.

More recent studies on new inference methods do not address the problem directly, either. For instance, the inference method proposed by Mazandarani and Xiu (2020) uses fractional horizontal membership functions. Zadeh's approach is a special case of this method for a fractional index equal to 1. Therefore, applying the conjunctive interpretation of if-then rules, the method is also vulnerable to the described problem. Similarly, an interesting mechanism of reasoning using moving membership functions is proposed Ho *et al.* (2010).

The state of the art analysis shows that the problem defined in Section 1 is still valid. Researchers are aware of the situation and propose to apply the logical approach in the cases where uncertainty is an important issue. However, as stated initially, all simplified solutions, even those using the implication-based inference, will produce incorrect results.

The structure of this article is as follows. Section 3 analyzes the uncertainty of the conjunctive approach at the level of classical logic. This stage allows defining the basics and propose solutions at the lowest possible level of complexity. The analysis is further generalized in Section 4 to fuzzy logic, where necessary definitions are provided, extending the theory of the inference based on a fuzzy truth value (Baldwin, 1979c). Section 5 focuses on the conjunctive inference process and, most importantly, provides solutions to the defined problem.

The final analysis is performed for two different approaches to fuzzy inference: the one presented by Baldwin (1979c), based on a fuzzy truth value, and the most popular and known, proposed by Zadeh (1975). To simplify names in further sections of this paper, both approaches to fuzzy inference will be referred to as Baldwin's inference and Zadeh's inference.

3. Uncertainty and the conjunctive approach in classical logic

The character of different types of uncertainty can be very well observed in the inference based on the truth values described by truth functions.

The truth functions in the context of fuzzy inference were presented by Bellman and Zadeh (1977). However, Baldwin (1979c) described a full inference approach, where analysis begins at the level of classical logic and is extended into fuzzy logic. As proved by Tong and Festathiou (1982), and also by Kudłacik and Łęski (Kudłacik and Łęski, 2021), Baldwin's solution can be directly transformed into Zadeh's compositional rule on inference (Zadeh, 1975). Therefore, the presented analysis is universal.

3.1. Uncertainty in two-valued logic. In classical, two-valued logic, Baldwin proposed defining the truth of sentences by the so-called truth functions ψ defined in two-valued, $\{0, 1\}$ space (Baldwin, 1979b; 1979c; 1979a).

Three truth functions were introduced and they correspond to the following situations: a proposition is true, false and undefined (undefined—when both logical truth and falsehood for given expression are equally possible). The functions are named accordingly ψ_{true},

ψ_{false}, ψ_{undef} and defined as follows (Baldwin, 1979c):

$$\psi_{true}(x) = \begin{cases} 1\,, & x=1,\\ 0\,, & x=0,\end{cases}$$

$$\psi_{false}(x) = \begin{cases} 0\,, & x=1,\\ 1\,, & x=0,\end{cases} \tag{4}$$

$$\psi_{undef}(x) = \begin{cases} 1\,, & x=1,\\ 1\,, & x=0,\end{cases}$$

where $x \in \{0,1\}$.

Baldwin (1979c) also defined obtaining truth functions of compound statements $p \wedge q$ and $p \vee q$, when the truth functions ψ_p and ψ_q of p and q clauses are known.

The definition of negation using truth functions, provided by Baldwin, creates the first step in investigation of the conjunctive approach. Due to an additional level, which is defining the truth of a sentence using a truth function, two forms of negation are acceptable and defined as follows (Baldwin, 1979c):

$$\neg\psi_p(x) = 1 - \psi_p(x), \tag{5}$$

$$\psi_{\neg p}(x) = \psi_p(1-x), \tag{6}$$

which corresponds to negation of a truth function and a truth function of a negated statement.

Table 1 (Baldwin, 1979c) shows the logical matrix for the negation (5) and (6). It should be noted that there is no corresponding truth function for negated undefined state. Baldwin did not develop such a case because it is useless from the inference in classical logic point of view. However, as the further sections of this article will prove, the case can be found useful.

Negation of the **undefined** state must also be undefined. Therefore, as a result, let us consider an additional truth function named 0-undef (read zero-undefined), marked as $\psi_{0\text{-}undef}$. The name refers to the character of the function, which is the special case of the undefined state and a value that it takes in the whole counter-domain.

Definition 1. Let $\psi_{0\text{-}undef}$ function represent also an undefined state and be described as follows:

$$\psi_{0\text{-}undef}(x) = \begin{cases} 0\,, & x=1,\\ 0\,, & x=0.\end{cases} \tag{7}$$

Table 1. Negation in Baldwin's approach for two-valued logic.

ψ_p	$\psi_{\neg p}$	$\neg\psi_p$
true	false	false
false	true	true
undefined	undefined	–

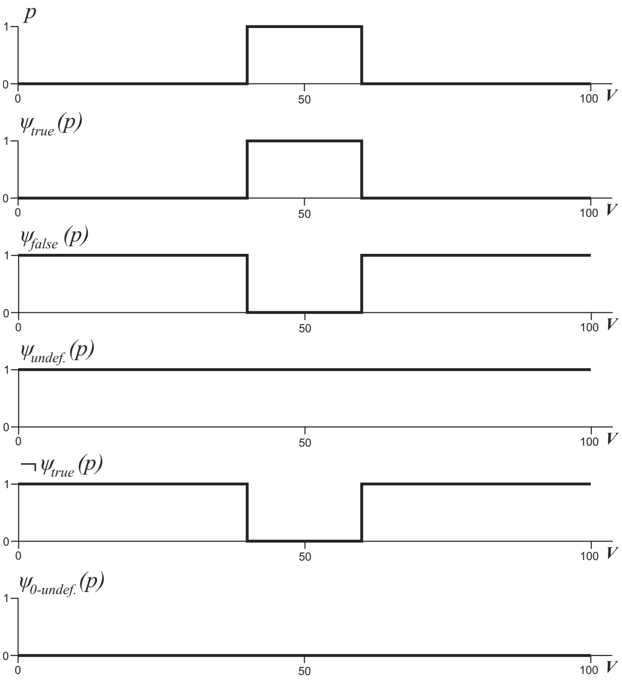

Fig. 3. Graphical interpretation of truth functions in two-valued logic. Subsequent graphs correspond with the logical value of a sentence p =*"A car is moving at moderate speed"* in the space of speed V defined in km/h. Modifications of logical values of a sentence using different truth functions are presented. Results for the proposed $\psi_{0\text{-}undef}$ are shown at the bottom.

It should be stressed that 0-undefined corresponds to some indefiniteness, but not in the logical sense. This means that a sentence described by the truth function $\psi_{0\text{-}undef}$ does not provide any information, such as ψ_{undef}. This is exactly the type of indefiniteness which can be found in systems with conjunctive interpretation of the *if-then* rule.

Figure 3 shows a graphical interpretation of different truth functions, also the newly proposed $\psi_{0\text{-}undef}$. The presented situation considers an example sentence p =*"A car is moving at moderate speed,"* assuming that the average car speed is between 40 and 60 km/h. Subsequent charts present logical values of a sentence p as well as its modification by assigning different truth functions to it for speed in space $V = [0, 100]$ defined in km/h.

According to (4) the statement *"A car is moving at moderate speed is true"*, i.e., $\psi_{true}(p)$, keeps input logical values of the sentence. Similarly, *"A car is moving at moderate speed is false"* changes the values stating that the car is moving at other than moderate speed (a value of 1.0 in the area other than the defined moderate speed). However, if the speed of the car is not known, it must be stated that *"A car is moving at moderate speed is undefined,"* therefore allowing all possible levels of speed to be true.

Figure 3 also shows an obvious negation *"it is not true that p,"* which means the same as marking the sentence p as false.

The last graph presents a modification of truth for the analyzed sentence using the $\psi_{\textit{0-undef}}$ function, which is the assignment of the *0-undefined* state. It can be said that, just as the truth $\psi_{\textit{undef}}(p)$, this statement does not give any information about the possible car speed. It is clear that from the classical logic viewpoint it is not correct, because in contrast to $\psi_{\textit{undef}}(p)$ it states that no other speed value is possible. However, this is the character of uncertainty in the conjunctive approach.

3.2. Conjunctive approach to inference in classical logic.

Baldwin (1979c) presented two schemes of inference based on modus ponendo ponens as well as modus tollendo tollens. Both the approaches use an implication $p \implies q$, because of the compliance with classical logic. In brief, the inference process is performed in the truth space, involving only the truth function of a premise and an implication. The inference result is a truth function of the conclusion, describing a fuzzy truth value of the conclusion (if the conclusion is true, false or undefined).

Replacing an implication with a conjunction operator in Baldwin's definitions allows us to create a new conjunctive inference model, which Baldwin did not take into account. The following theoretical analysis defines this new approach.

For two-valued logic, Baldwin's inference process based on a conjunction is very similar to the classical approach. Assuming the modus ponendo ponens, the process consists of several stages (Baldwin, 1979c). In the first one a truth function of a premise ($\psi_{\textit{true}}, \psi_{\textit{false}}, \psi_{\textit{undef}}$) is defined, which is nothing but stating that a premise is true, false or undefined.

The second stage generates the truth function of the conclusion by a max-min composition of a premise truth function with a conjunction matrix (instead of an implication matrix). The composition can be defined by the following equation:

$$\psi_q(y) = \max_{x\in\{0,1\}} \Big[\min\big(\psi_p(x),\, C(x,y)\big)\Big], \qquad (8)$$

where $C(x,y)$ represents a conjunction relation of the form

$$\begin{array}{cc|c} & & y \\ & & \begin{matrix} 0 & 1 \end{matrix} \\ x & \begin{matrix} 0 \\ 1 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array} = C(x,y). \qquad (9)$$

The composition (8) can be easily verified for all possible cases of input truth functions ψ:

$$\psi_{\textit{true}} \circ C = (0,1) \circ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = (0,1) = \psi_{\textit{true}},$$

$$\psi_{\textit{false}} \circ C = (1,0) \circ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = (0,0) = \psi_{\textit{0-undef}},$$

$$\psi_{\textit{undef}} \circ C = (1,1) \circ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = (0,1) = \psi_{\textit{true}},$$

The composition $\psi_{\textit{true}} \circ C$ corresponds to classical logic because the truth of a conclusion is obtained from the truth of a premise. However, the next two examples show significant differences. It can be stated that the composition $\psi_{\textit{false}} \circ C$ in a way corresponds to classical logic because its result gives $\psi_{\textit{0-undef}}$. As mentioned in Section 3.1, *0-undefined* can be interpreted as the undefined state in the conjunctive approach. Therefore, in this case, from the falsehood of a premise an indefiniteness of a conclusion (in a conjunctive sense) can be obtained.

The most divergent from the logic is the last example, in which the truth of a conclusion is obtained from a premise's indefiniteness. In this situation, the character of the conjunctive approach is revealed, which interprets only the compatibility of levels of membership functions on the side of the truth (the problem will be analyzed in detail as a part of the discussion concerning the conjunctive approach in fuzzy logic). Obviously, in two-valued logic, the desired indefiniteness can be obtained by using 0-undefined as an input, which will take the following form:

$$\psi_{\textit{0-undef}} \circ C = (0,0) \circ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = (0,0) = \psi_{\textit{0-undef}}.$$

Therefore, instead of the *undef* function for a premise, a *0-undef* one should be generated. Due to a highly limited number of possible results, obtaining the truth function of a premise is in this case very simple and can take the following form (membership functions μ_A and $\mu_{A'}$ correspond to the characteristic functions of classical sets and in this case can take only two values $\{0,1\}$):

$$\psi(\eta) = \sup_{\substack{x\in X \\ \eta=\mu_A(x)}} \Big[\mu_{A'}(x)\Big], \qquad (10)$$

and then is modified as

$$\psi'(\eta) = \psi(\eta) - \big(\psi(0) \cdot \psi(1)\big). \tag{11}$$

This operation will guarantee obtaining 0-undefined ($\psi_{\textit{0-undef}}$) in case of indefiniteness ($\psi_{\textit{undef}}$). Other types of truth functions will remain unchanged.

Taking into account only appropriate results of the inference process shown above, another solution is possible. The undefined state of a conclusion is obtained in two cases: where the truth function of a premise is equal to $\psi_{\textit{false}}$ and $\psi_{\textit{0-undef}}$. In both situations $\psi_{\textit{false}}(0) = \psi_{\textit{0-undef}}(0) = 1$. Therefore, if the truth function for 0 (the side of falsehood) gives 1 as a result, it can be modified to $\psi_{\textit{0-undef}}$ as follows:

$$\psi'(\eta) = \min\big[\psi(\eta), 1 - \psi(0)\big]. \tag{12}$$

Only $\psi_{\textit{true}}$ will remain unchanged. This simplification is very important from the fuzzy analysis viewpoint, because the idea can be directly generalized in fuzzy logic, which is described in what follows.

The process of transforming the undefined state into 0-undefined for the conjunctive approach is a one-time procedure. After the transformation, other phases of Baldwin's inference, which is obtaining a truth function of conclusion and truth functional modification, remain unchanged.

4. Conjunctive indefiniteness in fuzzy logic

A fuzzy truth value was defined by Bellman and Zadeh (1977). The scientists considered such statements to be

$$\mathbf{x} \text{ is } A \text{ is } \tau, \tag{13}$$

where τ is a fuzzy set corresponding to linguistic description of a statement's truth. The set τ is a fuzzy restriction of values that can be assigned to A (Bellman and Zadeh, 1977).

The statement (13) is equivalent to

$$\mathbf{x} \text{ is } U, \tag{14}$$

where U is a fuzzy set and its membership function is obtained by the truth functional modification (Bellman and Zadeh, 1977),

$$\mu_U(x) = \tau\big[\mu_A(x)\big]. \tag{15}$$

Infinite-valued logic allows defining an infinite number of different τ functions. Extending deliberations for classical logic, Baldwin defined and named several basic fuzzy truth values like **very true**, **very false**, **fairly true**, **fairly false**, **absolutely true**, **absolutely false** and the negation procedure (Baldwin, 1979c).

In the context of the presented analysis for truth functions in classical logic the above-mentioned set of functions must be completed with the **0-undefined**, described by $\tau_{\textit{0-undef}}$ function.

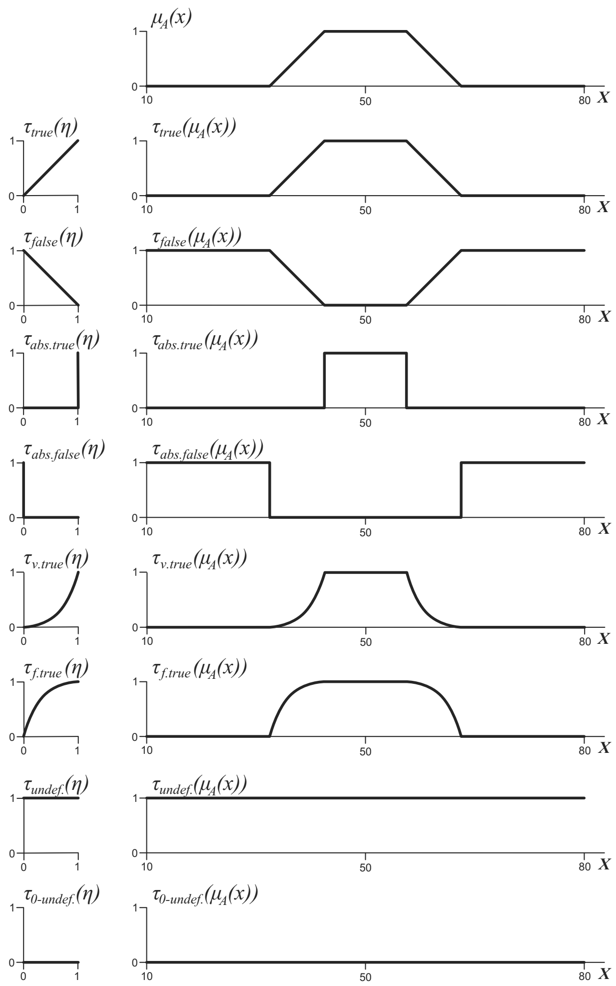

Fig. 4. Graphical interpretation of truth functions in fuzzy logic. Subsequent graphs show a truth functional modification of a set A (*"moderate speed"*) by different fuzzy truth values τ. Results for the proposed $\tau_{\textit{0-undef}}$ are presented at the bottom.

Definition 2. Let the $\tau_{\textit{0-undef}}$ function represent another undefined state and be described as follows:

$$\underset{\eta\in[0,1]}{\forall} \quad \tau_{\textit{0-undef}}(\eta) = 0. \tag{16}$$

Figure 4 shows a graphical interpretation of different τ functions in fuzzy logic, where also results for the proposed $\tau_{\textit{0-undef}}$ are presented. In the upper part a membership function of a sample A set is shown and it corresponds to linguistic definition *"moderate speed"* of some moving object.

Similarly to the example in Fig. 3 for classical logic, space X presents the possible speed given in km/h. On the left-hand side of the figure there are graphs presenting the chosen the truth functions τ. However, on the right-hand side there are graphs presenting truth functional modification of a set A by these functions.

Therefore, the illustrated modifications present assigning a given truth τ to the linguistic expression *"moderate speed."*

The first graph below the function μ_A corresponds to the statement *"an object is moving at moderate speed is **true**."* A function τ_{true} preserves input truth values, therefore the graph remains unchanged. Similarly, assigning to the speed of the object the statement *"is **false**"* generates the complement set to A, which correctly suggests that the speed is different than moderate in this case.

Further, the meaning of other functions like **absolutely true** and **absolutely false** or the influence of **very** and **fairly** can be similarly observed.

The last two examples show how truth functions work for the undefined state. A function τ_{undef} is compliant with an interpretation of indefiniteness in a logical sense allowing in this case equal possibility of occurrence of all speed values from the space X. However, the function $\tau_{0\text{-}undef}$, representing the **0-undefined** state, incompatible with the classical logic, is used in systems with the conjunctive interpretation of the *if-then* rule.

The subject of fuzzy truth values was also extensively covered by Dubois and Prade. They considered other useful kinds of fuzzy truth values such as **at least α-true** and **at most α-false** (Dubois and Prade, 1999).

4.1. Obtaining a truth function in fuzzy logic. The infinite variety of possible truth functions τ does not allow one to arbitrarily choose a truth function. Baldwin (1979c) proposed reversing an operation of truth functional modification (an inverse truth functional modification—ITFM) (15) in order to obtain a set τ on the basis of a fact an a premise (A' and A):

$$\underset{\eta\in[0,1]}{\forall} \quad \tau(\eta) = \sup_{\substack{x\in X\\ \eta=\mu_A(x)}} \left[\mu_{A'}(x)\right], \tag{17}$$

where τ defines a truth function of "**x** is A", which in a way represents how a linguistic variable **x** corresponds to a description in the form of a set A for a given fact "**x** is A'."

It is worth noting that the expression (17) can be obtained using the extension principle by extending the operation of calculating the value of a membership function μ_A, where the fuzzy set A' is given instead of an input numerical value.

The general concept of the ITFM presented above can be briefly described as a mechanism of creating τ depending on the compatibility of sets A and A'. Generated truth functions take forms from the **absolutely true** for the highest compatibility, through **very true**, **fairly true**, **fairly false** for less compatibility, to the **absolutely false** for the complete lack of compatibility.

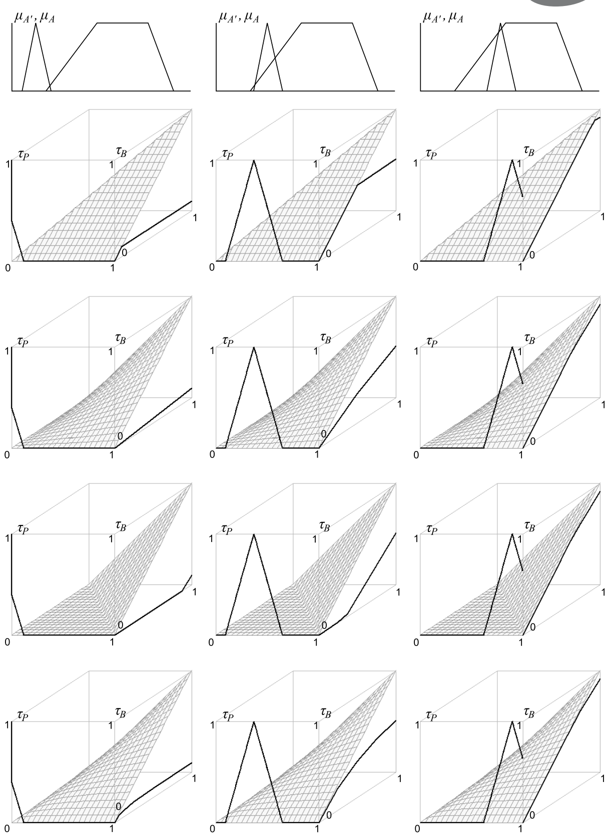

Fig. 5. Obtaining the premise and conclusion truth functions in the conjunctive interpretation of the *if-then* rule using four different T-norms (from the top: minimum, algebraic, Łukasiewicz and Hamacher).

The process can be observed in Fig. 5 for three different situations presented in subsequent columns. The figure also presents obtaining a truth function of the conclusion; however, the truth function of the premise, showing the fact-premise compatibility, is also clearly visible. It is marked as τ_P and drawn in the front plane of all 3D charts. For the purposes of this analysis only the first two rows of charts are sufficient.

The first situation (first column) shows very low fact-premise compatibility ($\mu_{A'}$ description almost outside of μ_A), therefore the resulting truth function τ_P indicates almost absolutely false according to Baldwin's definition (see $\tau_{abs.false}$ in Fig. 4), because it is not true that A is A'.

The second situation shows higher compatibility (a whole description $\mu_{A'}$ within the slope of μ_A), which becomes very large in the third situation (τ_P can be described nearly as very true according to Baldwin's definition—see $\tau_{v.true}$ in Fig. 4).

5. Fuzzy inference in a conjunctive interpretation of the *if-then* rule

As previously mentioned, Baldwin did not consider a conjunctive solution in his approach to fuzzy inference. However, the general mechanism will not change (Baldwin, 1979c).

The process consists of three stages. First of all a truth function of a premise τ_P have to be obtained according to (17). If there is a compound premise in the rule, a compound truth function has to be obtained depending on join operations. Baldwin (1979c) defined this procedure precisely in his work and it is not important from this analysis point of view. Compound or not, the general meaning of τ_P is the same. This one function describes the compatibility of facts and premises in a rule.

Then the process can proceed to the second stage, which is obtaining a truth function of conclusion τ_B on the basis of the previously obtained premise truth function τ_P. It can be performed by generalizing the max-min composition (8) presented for two-valued logic into sup-T-norm composition and considering the conjunctive approach (T-norm instead of an implication). Therefore, the expression (8) will take the following form of supremum-T-norm:

$$\underset{\phi\in[0,1]}{\forall} \tau_B(\phi) = \sup_{\eta\in[0,1]} \Big[\tau_P(\eta) \star_{T_1} T_2(\eta,\phi)\Big], \tag{18}$$

where $T_2()$ represents any T-norm defining the conjunctive approach. It should be noted that the logical approach differs only in using an implication instead of the T_2 T-norm. Marking both T-norms in similar manner, Eqn. (18) can be presented as follows:

$$\underset{\phi\in[0,1]}{\forall} \tau_B(\phi) = \sup_{\eta\in[0,1]} \Big[\tau_P(\eta) \star_{T_1} \eta \star_{T_2} \phi\Big]. \tag{19}$$

It is worth mentioning that all the components of the statement in the square brackets are joined by T-norms, which, due to commutativity, can be used in any order. Obviously, the T-norms can be the same or different.

The last step of Baldwin's inference is obtaining a fuzzy result on the basis of a truth function of conclusion (Baldwin, 1979c). This stage does not differ from the approach with a logical interpretation of the *if-then* rule and, as shown in the previous section, a fuzzy result B' is obtained by a truth functional modification according to (15) using τ_B (the truth functional modification was also shown in Fig. 4).

Figure 5 visualizes the process of obtaining a τ_B function by the expression (19). The top row presents three sample situations, in which membership functions $\mu_{A'}$ and μ_A of three facts and premises are depicted in different configurations. In each situation, in four rows, the 3D charts were added, presenting the operation (19) for four different T-norms (subsequently: minimum, algebraic, Łukasiewicz and Hamacher). In every 3D chart both τ_P and τ_B are marked with a bold line. The function τ_P is obtained according to (17) based on a premise and a fact.

It can be noticed how the output truth function τ_B changes in the subsequent columns from a form close to the **0-undefined** to a form close to the **very true** together with increasing level of compatibility of facts and premises.

Figure 5 does not show a situation in which based on a premise and a fact, the **undefined** state will be obtained ($\tau_P = \tau_{\mathit{undef}}$), i.e., when a truth function equals 1 in the whole domain. This kind of function is obtained when a fact is much less precise and within its core an includes such interval of space in which all the possible values of a membership function of a premise are located. Such an example is presented in Fig. 6. The first two graphs show sample membership functions of a fact and a premise, whereas the last graph contains $\tau_P = \tau_{\mathit{undef}}$ obtained in these cases. In the first situation the core of $\mu_{A'}$ contains only the rising slope of μ_A, and in the second one the whole interval, in which $\mu_A \neq 0$.

As has been shown for two-valued logic, the composition of ψ_{undef} with a conjunction matrix gives ψ_{true} as the result. Analyzing 3D graphs in Fig. 5 it can be observed that also in fuzzy logic the composition of τ_{undef} with T-norm will generate $\tau_B = \tau_{\mathit{true}}$. Such an operation results directly from the character of the solution or from using T-norm, to be precise. In the conjunctive approach, the larger the intersection of a premise and a fact, the higher their compatibility. In the case of **indefiniteness**, as in Fig. 6, the intersections represent significant areas, which results in the highest compatibility. Therefore, the conjunctive approach works correctly only for facts defined more precisely than a premise (facts with relatively “narrow core”).

The same effect can be noticed in the case of Zadeh's compositional rule of inference. Equation (20) presents obtaining a membership function of a result $\mu_{B'}$ for the conjunctive approach (Czogała and Łęski, 2000):

$$\underset{y\in Y}{\forall} \mu_{B'}(y) = \sup_{x\in X} \left[\mu_{A'}(x) \star_T \mu_A(x) \star_T \mu_B(y)\right]. \tag{20}$$

It can be noticed that the higher compatibility of a fact and a premise depends only on the intersection $\mu_{A'}(x) \star_T \mu_A(x)$. This approach does not differentiate the **undefined** and the **true** states, either. In both the cases the supremum of the intersection of a fact and a premise is equal to 1, which is the highest possible level.

6. Modification of the conjunctive approach

The presented analysis has shown that the conjunctive approach does not distinguish the **undefined** state, generating the exact same result as in the case of

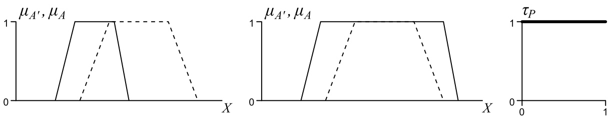

Fig. 6. Sample situations, in which $\tau_P = \tau_{undef}$ is obtained. Membership functions of facts are marked with a solid line, while membership functions of premises with a dashed line.

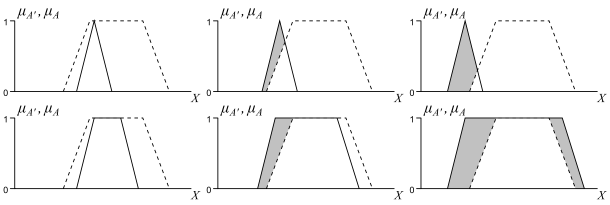

Fig. 7. Common features of incompatibility of a fact and a premise for the **false** and the **undefined** states. The facts (A') are marked with a solid line, whereas the premises (A) with a dashed line. The gray color shows the difference of areas $P_{A'} - P_{A' \cap A}$, where $P_{A'}$ and $P_{A' \cap A}$ represent an area under the membership function of a fact and intersection of a fact and a premise, respectively.

true. Section 3.2 describes a solution for two-valued logic, which is conversion of the undefined (ψ_{undef}) into 0-undefined ($\psi_{0\text{-}undef}$). In this section a generalization of this procedure will be proposed.

Fuzzy logic offers a possibility to create an infinite number of different truth functions; therefore, suggesting an appropriate modification of obtaining τ_P is more complex. In the context of fuzzy inference an appropriate form of the conclusion truth function (τ_B) should be considered: **true** (function τ_{true}) and **0-undefined** (function $\tau_{0\text{-}undef}$). The **true** state of a conclusion should be obtained only with an adequate compatibility of a fact and a premise, whereas **0-undefined**—when there is no compatibility. However, the lack of compatibility should appear when the premise is **false** and when the truth is **undefined**, which for **undefined** is not provided by the conjunctive approach. That exact situation could be also observed in analysis for two-valued logic.

Figure 7 allows noting common features of the **false** and the **undefined** states. Consider it as a starting point for designing a relevant modification of the method obtaining a truth function of a premise. The upper row of the graphs shows a situation in which a fact is being more and more moved out of a premise range defining its increasing **falsehood** according to Baldwin's definition, and therefore decreasing the compatibility of A' with A. The lower row of the graphs also shows the decreasing compatibility of a fact and a premise. However, this time this is done by increasing the core of a fuzzy set A' and obtaining larger **indefiniteness**. In both cases the gray color shows the difference between the area created by a membership function of a fact and the area of intersection of a fact and a premise. It can be seen that the size of this area is inversely related to he compatibility of a fact and a premise. This dependence can be used to obtain a proper truth function in the conjunctive approach,

$$\underset{\eta\in[0,1]}{\forall} \tau_P(\eta) = \sup_{\substack{x\in X\\ \eta=\mu_A(x)}} \Big[\mu_{A'}(x)\Big] \star_T \left(1 - \frac{P_{A'} - P_{A'\cap A}}{P_{A'}}\right), \tag{21}$$

where $P_{A'}$ represents the area under the membership function of a fact and $P_{A'\cap A}$ is the area formed by the intersection of membership functions of a fact and a premise. Therefore, it can be said that the values of a truth function obtained by Baldwin's approach according to (17) are reduced by the compatibility of a fact and a premise defined by the areas. When the intersection area of both membership functions equals zero (**absolutely false** in Baldwin's sense), this equation will generate the **0-undefined** state because its second part will be equal to 0. In the conjunctive approach, a truth function of a premise $\tau_P = \tau_{0\text{-}undef}$ generates obviously the **0-undefined** state of a conclusion, which is a correct result.

When the area under a premise membership function comprises the area under a fact membership function (all situations between the **true** and the **absolutely true** in Baldwin's sense), suggested modification will not bring any changes because $P_{A'} - P_{A'\cap A} = 0$. Therefore, the second part of Eqn. (21) will be equal to 1. The last case depicts comprising an ever greater area under a premise membership function by the fact membership function, as shown before for a sample situation in the second row of Fig. 7. In this case the **undefined** state of a premise is obtained in Baldwin's sense. The suggested Eqn. (21) will diminish the values of τ_P depending on the ratio of the $P_{A'}$ and $P_{A'\cap A}$ areas. When the area under a fact membership function is twice as big as that under a premise membership function, the compatibility will be equal to $1/2$. Therefore, in this case, $\tau_P = 1/2$ in the whole domain, because $\tau_{undef} - 1/2 = 1/2$. Of course, together with extending the area under a fact membership function, lower values of output function τ_P will be obtained. Referring to the terminology adapted in this paper, an infinite number of **undefined** states can be marked as $\tau_{\alpha\text{-}undef}$.

Definition 3. Let $\tau_{\alpha\text{-}undef}$ represent an infinite number of undefined states, depending on $\alpha \in [0,1]$, and let it be described as follows:

$$\underset{\eta\in[0,1]}{\forall} \tau_{\alpha\text{-}undef}(\eta) = \alpha. \tag{22}$$

As long as applying Eqn. (21) significantly modifies the behavior of the conjunctive approach, it is not perfect because for the **undefined** state of a premise in Baldwin's sense the **0-undefined** state will never be obtained (only **α-undefined** at the level of α, which depends on the ratio of the areas). An operation (21) in this case can still be improved by a significant decrease in α. It can be obtained by raising the value of the part defining compatibility to an arbitrary power K. Then (21) will take the following form:

$$\tau_P(\eta) = \sup_{\substack{x\in X \\ \eta=\mu_A(x)}} \Big[\mu_{A'}(x)\Big] \star_T \left(1 - \frac{P_{A'} - P_{A'\cap A}}{P_{A'}}\right)^K . \tag{23}$$

The coefficient K can be interpreted as a level of compatibility reduction in relation to increasing the ratio of the areas.

Allowing the mechanism to obtain $\tau_P = \tau_{\textit{0-undef}}$ in the case of the **undefined** state of a premise requires a slightly different approach. Instead of focusing on the ratio of fields, one can consider the compatibility of a fact with a premise as the maximum difference $\mu_{A'} - \mu_A$, which is in a sense the height of a figure defined by $P_{A'} - P_{A'\cap A}$. In such a case compatibility is inversely related to the maximum difference $\mu_{A'} - \mu_A$. Therefore, τ_P can take the following form:

$$\begin{aligned}\tau_P(\eta) = & \sup_{\substack{x\in X \\ \eta=\mu_A(x)}} \Big[\mu_{A'}(x)\Big] \\ & \star_T \Big\{1 - \sup_{x\in X} \Big[0,\ \mu_{A'}(x) - \mu_A(x)\Big]\Big\}.\end{aligned} \tag{24}$$

This solution is characterized by fast convergence with extreme levels of compatibility. When a premise area totally comprises a fact area, the second part of the equation will be equal to 1, because the area $P_{A'} - P_{A'\cap A}$ equals zero. In such a way all created τ_P functions between the **true** and the **absolutely true** are not modified. However, when compatibility becomes lower (by moving toward the **false** or the **undefined** states of a premise), the maximum difference of $\mu_{A'} - \mu_A$ will also increase. The highest level of difference that equals 1 will be obtained for the **undefined** or at least the **false** states of a premise (when the maximum value of a fact membership function will be contained within the range where a premise membership function equals 0, thus $\mu_{A'} - \mu_A = 1$).

An idea almost identical to the one presented by (24) can be also realized by direct generalization of (12), which for fuzzy logic can take the following form:

$$\tau'_P(\eta) = \tau_P(\eta) \star_T (1 - \tau_P(0)) . \tag{25}$$

In this approach the modified truth function τ'_P is obtained on the basis of the original τ_P, given by (17). The results of this solution are very similar to (24), which allows the system to obtain the 0-undefined state when a maximum value of a fact membership function, which is 1, will be located within the area where a premise membership function equals 0.

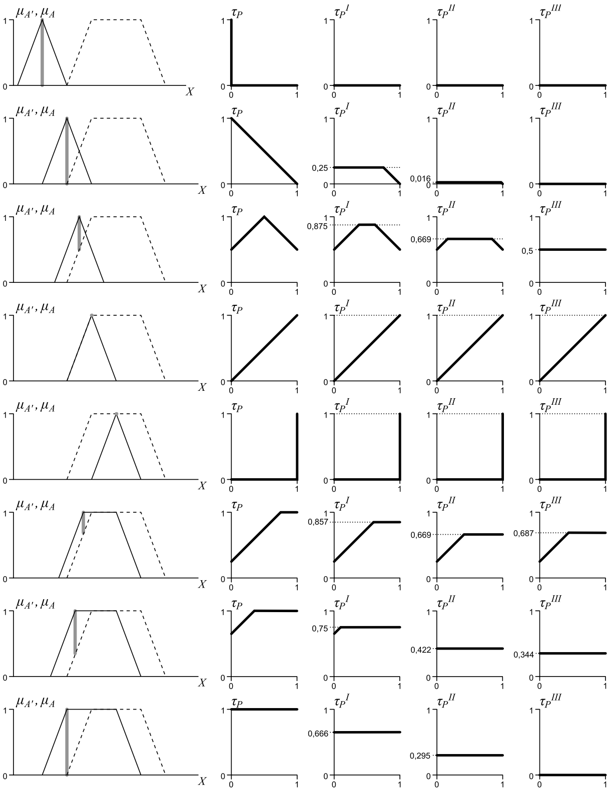

Fig. 8. Examples of different modifications of obtaining the truth function of a premise. For every situation shown in the charts on the left, four truth functions have been obtained using different methods. Fact membership functions ($\mu_{A'}$) are shown with a solid line, whereas functions of premises (μ_A) with a dashed line. The thick, gray line indicates the maximum difference $\mu_{A'} - \mu_A$. The τ_P functions were obtained without any modification, $\tau_P{}^I$ was obtained by (21), $\tau_P{}^{II}$ by (23) for $K = 3$, whereas $\tau_P{}^{III}$ by (24) or (25).

Figure 8 shows an impact of the modifications (21), (23) and (24) or (25) on obtaining a truth function of a premise in a few characteristic situations. The left-hand side illustrates the charts of membership functions and the right-hand side shows for these cases the truth functions of a premise obtained using different modifications. Functions τ_P were obtained according to (17), which means no changes and will be used as a reference.

The next functions, $\tau_P{}^I$, $\tau_P{}^{II}$ and $\tau_P{}^{III}$, were obtained according to (21), (23) and (24), respectively (for presented examples Eqn. (25) gives the same results as (24)). In the charts containing facts and premises the gray line indicates the maximum difference $\mu_{A'} - \mu_A$, therefore, its direct impact on obtaining ${\tau_P}^{III}$ can be observed.

It can be noticed how the implemented modifications change the form of falsehood and indefiniteness by being reduced down toward **0-undefined** to a bigger or smaller extent (depending on the method used). However, all the states between the **true** and the **absolutely true** remain unchanged. In this way, by choosing and adjusting a proper modification method, the desired impact of indefiniteness on the inference result can be controlled.

To summarize the analysis for the proposed relationships (21), (23), (24) and (25), it must be emphasized that the modifications of the truth function influences the cases for which there is at least some level of incompatibility between a given fact and a premise. As described at the beginning of the section, a certain level of incompatibility is obtained for cases when the relation between a fact an a premise, reflected by their truth function, moves more and more into the false or the undefined states. Therefore, it can be stated that all solutions represent different forms of generalization of (12) proposed for two-valued logic. From the practical point of view, the difference between the proposed modifications exists only in the speed of approaching the 0-undefined state by the result, based on the level of incompatibility.

6.0.1. Modification in the context of the compositional rule of inference. A similar modification of the conjunctive approach can be used in the case of Zadeh's compositional rule of inference. Considering subsequent changes in (21), (23) and (24), the statement (20) takes respectively the following forms:

$$\mu_{B'}(y) = \sup_{x\in X} \left[\mu_{A'}(x) \star_T \mu_A(x) \star_T \mu_B(y)\right] \star_T \star_T \left(1 - \frac{P_{A'} - P_{A'\cap A}}{P_{A'}}\right), \tag{26}$$

$$\mu_{B'}(y) = \sup_{x\in X} \left[\mu_{A'}(x) \star_T \mu_A(x) \star_T \mu_B(y)\right] \star_T \star_T \left(1 - \frac{P_{A'} - P_{A'\cap A}}{P_{A'}}\right)^K, \tag{27}$$

and

$$\mu_{B'}(y) = \sup_{x\in X} \left[\mu_{A'}(x) \star_T \mu_A(x) \star_T \mu_B(y)\right] \star_T \star_T \left\{1 - \sup_{x\in X} \left[0,\ \mu_{A'}(x) - \mu_A(x)\right]\right\}. \tag{28}$$

In this case the modifications decrease directly the values of the obtained conclusion membership function according to a defined level of compatibility.

Obviously, the solution (25) cannot be presented under these circumstances, because there is no truth function to modify.

6.0.2. Modifications in the context of a compound premise. The previous analysis concerned modifications of obtaining a truth function for a simple premise, which is not compound premise. When a compound premise is considered, the easiest solution is to modify only one output truth function after composition. Thus, in this situation, obtaining particular truth functions, as well as their composition, will not change. In this case, Baldwin's equations for truth function composition can be used (Baldwin, 1979c).

Moreover, for each individual premise, the Z coefficient, representing the level of compatibility with a fact, must be obtained according to the chosen method (based on the relation between areas or the difference). Therefore, the Z coefficient can take one of the following forms:

$$Z = \left(1 - \frac{P_{A'} - P_{A'\cap A}}{P_{A'}}\right), \tag{29}$$

$$Z = \left(1 - \frac{P_{A'} - P_{A'\cap A}}{P_{A'}}\right)^K \tag{30}$$

and

$$Z = \left\{1 - \sup_{x\in X} \left[0,\ \mu_{A'}(x) - \mu_A(x)\right]\right\}, \tag{31}$$

or

$$Z = \left(1 - \tau_P(0)\right). \tag{32}$$

Next, from the obtained levels of compatibility Z, one compound level should be achieved using any T-norm, when the premises have been joined by the "and" conjunction, or any S-norm, when the premises have been joined by "or", which will take the following forms:

$$Z = Z_{A_1} \star_T Z_{A_2} \tag{33}$$

or

$$Z = Z_{A_1} \star_S Z_{A_2}, \tag{34}$$

where Z_{A_1} and Z_{A_2} correspond to given levels of compatibility obtained for premises A_1 and A_2, respectively.

The last step involves using the obtained compound level of compatibility to modify the compound truth function

$${\tau_P}'(\eta) = \tau_P(\eta) \star_T Z. \tag{35}$$

The defined coefficient Z can also be used to obtain a conclusion by the compositional rule of inference for the conjunctive approach. The compositional rule of inference in this case for two premises takes the following form:

$$\mu_{B'}(y) = \sup_{\substack{x_1\in X_1\\ x_2\in X_2}} \left[\mu_{A_1'}(x_1) \star_T \mu_{A_2'}(x_2) \star_T \mu_{A_1}(x_1) \star_T \star_T\ \mu_{A_2}(x_2) \star_T\ \mu_B(y)\right] \star_T Z. \tag{36}$$

The presented solutions can be generalized for a premise consisting of any number of simple premises.

In Baldwin's approach the output truth function can be obtained from the extension principle or by a composition of several functions (Baldwin, 1979c). Similarly the composition of any number of individual Z coefficients can be obtained by (33) or (34).

A solution using the compositional rule of inference for a larger number of premises will not bring any changes in the formula. The obtained result can be modified using compound Z coefficient, just as shown in (36) for two premises.

7. Practical example

This section will demonstrate the effect of the proposed modifications for several sample situations. In order to facilitate the analysis of the example, the inference system is assumed to be based on a single rule with a single premise and conclusion in the following form:

$$\begin{array}{ll} \text{FACT :} & X \text{ is } A' \\ \text{RULE :} & \text{if } X \text{ is } A \text{ then } Y \text{ is } B \\ \hline \text{CONCLUSION :} & Y \text{ is } B'. \end{array} \tag{37}$$

The inference result B' will be determined in three ways to illustrate the differences and show the advantages of the proposed solution. For ease of analysis, the inference example uses the Mamdani–Assilan system, with the minimum function as T-norm. However, it should be emphasized that the proposed approach is universal by appropriately modifying the activation level of the rule (determining the level of compatibility of the fact with the premise). Hence, it is possible to apply it in any fuzzy inference solution.

Figure 9 presents the inference results for several example situations. The first row of the graphs shows the form of the membership function for the premise μ_A in the domain X and the conclusion μ_B in the domain Y. The next rows of the graphs show particular situations of compatibility between a fact and a premise in the space X and results $\mu_{B'}$, $\mu_{B''}$ and $\mu_{B'''}$ in the domain Y. The three results are determined by different means. The membership function $\mu_{B'}$ is obtained classically in the conjunctive approach without any modification. Let this be the reference result. The next two functions, $\mu_{B''}$ and $\mu_{B'''}$, are determined using the modifications (29) (with field difference) and (31) (with maximum difference of membership functions), respectively.

The sample situations are presented in two groups. In the first, one the fact membership function has triangular form. In the second one, it is trapezoidal, and one can observe the influence of uncertainty in the modified solutions.

When the fact membership function has a triangular form, the proposed modifications correctly do not affect the obtained result significantly. The modification based

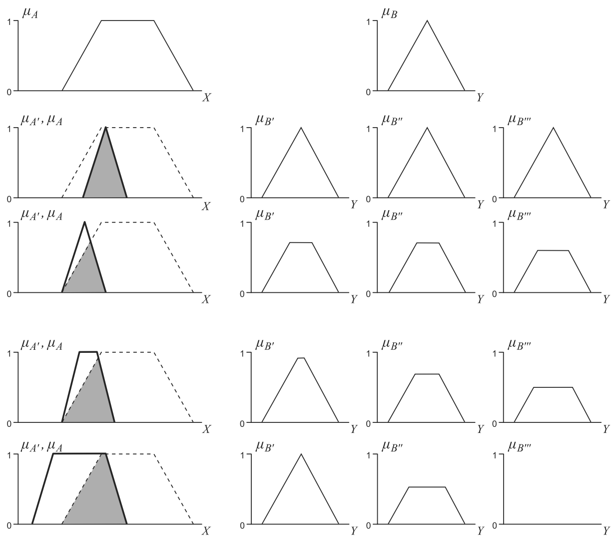

Fig. 9. Influence of the proposed modifications on the obtained results for several sample situations.

on the difference of fields gives identical results in this situation (result $\mu_{B''}$). Only the result $\mu_{B'''}$ is different because it uses the maximum difference-based modification, which is characterized by the fastest convergence to the extreme levels of compatibility.

When the fact membership function has trapezoidal form (higher uncertainty model), one can see the impact of the proposed modifications, in order to obtain the correct result. The function $\mu_{B'}$ in both latter cases will take into account the maximum level of intersection between the fact and the premise (gray area). Hence, the results in these situations will indicate a high compatibility and thus a high activation level of the rule. The proposed modifications, on the other hand, will adequately consider greater indeterminacy and hence greater uncertainty in the final conclusion.

The examples do not present only the modification (30), but its result, depending on the given K, will lie between $\mu_{B''}$ and $\mu_{B'''}$. This allows us to determine the effect of uncertainty on the final result.

8. Conclusion

To summarize the discussion presented in this paper, the following two areas are particularly worth emphasizing. First of all, using Baldwin's approach based on a fuzzy truth value, we described the process of approximate inference employing the conjunctive interpretation of the *if-then* rule in two-valued logic (8) and fuzzy logic (19). Such an approach for Baldwin's inference has never been described in the literature.

The presented analysis for classical and fuzzy logic allowed us to define additional truth functions (Definitions 1 and 2 of the **0-undefined** state). The functions naturally occur in the conjunctive approach, filling the gap in a wide range of various truth functions considered by Bellman and Zadeh (1977), Baldwin (1979c) as well as Dubois and Prade (1999; 1996). Moreover, the deliberations showed the usefulness of many other kinds of undefined states described as **α-undefined** (Definition 3). The **α-undefined** state can be considered a generalization of the **0-undefined**.

The second area focused on extensions to the conjunctive approach for Baldwin's and Zadeh's fuzzy inference allowing conjunctive systems to interpret the undefined state adequately, not generating a true conclusion for an undefined premise. In a way, the extensions make the conjunctive approach similar to the classical one, where an undefined premise implies an undefined conclusion. However, it must be emphasized that most practical implementations are based on simplified approaches, where the relationship between a fact and a premise is mapped into one value in the $[0, 1]$ range. Therefore, under such circumstances also simplified logical approaches can benefit from the proposed solutions.

It must be emphasized that our intention was not to create a new form of logic but to provide a solution to an existing problem, which is intentionally or unintentionally ignored. In our opinion, developers of fuzzy systems encountering the problem of undefined premises should use the logical approach, wherever possible, allowing interpretation of all premise truth values (the true, the false, and the undefined). Unfortunately, most computationally efficient systems use a simplified approach. Mapping the fact–premise relationship to only one truth value, using the highest result of an intersection between A' and A (like in widespread applications proposed by Mamdani and Assilan or Takagi, Sugeno, and Kang). Therefore, there is no way to distinguish the situation when, e.g., $A' \in A$ and $A \in A'$, where the latter should generate the undefined (cf., e.g., Fig. 7). The proposed modifications (21), (23), (24), (25) solve that problem in Baldwin's approach and (26), (27), (28) in Zadeh's approach.

It is also important to emphasize that the described extensions are unnecessary for systems where indefiniteness is not encountered; for instance, fuzzification of input variables using a singleton, or generally, situations where facts are defined much more precisely than premises (cores of facts' fuzzy sets are relatively smaller than cores of premises' fuzzy sets). This is due to the fact that the indefiniteness of the premise in Baldwin's sense will never be obtained.

On the other hand, the proposed solutions can be beneficial when an output of one fuzzy system is used as an input of another one, or definitions of fuzzy sets are automatically generated. Then, the fuzziness of the obtained facts can be large in comparison with a premise.

References

Azzini, A., Marrara, S., Sassi, R. and Scotti, F. (2008). A fuzzy approach to multimodal biometric continuous authentication, *Fuzzy Optimization and Decision Making* **7**(243): 243–256.

Baldwin, J. (1979a). *Advances in Fuzzy Set Theory and Applications*, North-Holland, Amsterdam, pp. 93–115.

Baldwin, J. (1979b). Fuzzy logic and fuzzy reasoning, *International Journal of Man-Machine Studies* **11**(4): 465–480.

Baldwin, J. (1979c). A new approach to approximate reasoning using a fuzzy logic, *Fuzzy Sets and Systems* **2**(4): 309–325.

Bellman, R. and Zadeh, L. (1977). *Modern Uses of Multiple-Valued Logic. Episteme*, Springer, Dordrecht, pp. 103–165.

Cordon, O., Herrera, F. and Peregrin, A. (1997). Applicability of the fuzzy operators in the design of fuzzy logic controllers, *Fuzzy Sets and Systems* **86**(1): 15–41.

Czabanski, R., Jezewski, M. and Leski, J. (2017). *Introduction to Fuzzy Systems*, Springer, Cham, pp. 23–43.

Czogała, E. and Kowalczyk, R. (1996). Investigation of selected fuzzy operations and implications for engineering, *IEEE 5th International Conference Fuzzy Systems, New Orleans, USA*, pp. 879–885.

Czogała, E. and Łęski, J. (2000). *Fuzzy and Neuro-Fuzzy Intelligent Systems*, Physica, Springer-Verlag, Heidelberg.

Czogała, E. and Łęski, J. (2001). On equivalence of approximate reasoning results using different interpretations of if-then rules, *Fuzzy Sets and Systems* **117**(2): 279–296.

Dubois, D. and Prade, H. (1999). Fuzzy sets in approximate reasoning. Part 1: Inference with possibility distribution, *Fuzzy Sets and Systems* **100**(Supp. 1): 73–132.

Dubois, D. and Prade, H. (1996). What are fuzzy rules and how to use them, *Fuzzy Sets and Systems* **84**(2): 169–185.

Grzegorzewski, P., Hryniewicz, O. and Romaniuk, M. (2020). Flexible resampling for fuzzy data, *International Journal of Applied Mathematics and Computer Science* **30**(2): 281–297, DOI: 10.34768/amcs-2020-0022.

Ho, C., Li, J. and Gwak, S. (2010). Research of a new fuzzy reasoning method by moving of fuzzy membership functions, *2010 International Symposium on Intelligence Information Processing and Trusted Computing, Huanggang, China*, pp. 297–300.

Izquierdo, S.S. and Izquierdo, L.R. (2018). Mamdani fuzzy systems for modelling and simulation: A critical assessment, *Journal of Artificial Societies and Social Simulation* **21**(3): 2.

Klir, G.J., Clair, U.S. and Yuan, B. (1997). *Fuzzy Set Theory: Foundations and Applications*, Prentice Hall, Upper Saddle River.

Kudłacik, P. (2010). Advantages of an approximate reasoning based on a fuzzy truth value, *Medical Informatics & Technologies* **16**: 125–132.

Kudłacik, P. (2012). Performance evaluation of Baldwin's fuzzy reasoning for large knowledge bases, *Medical Informatics & Technologies* **20**: 29–38.

Kudłacik, P. (2013). An analysis of using triangular truth function in fuzzy reasoning based on a fuzzy truth value, *Medical Informatics & Technologies* **22**: 103–110.

Kudłacik, P. and Łęski, J. (2021). Practical aspects of equivalence of Baldwin's and Zadeh's fuzzy inference, *Journal of Intelligent & Fuzzy Systems* **40**(3): 4617–4636.

Mamdani, E. and Assilan, S. (1975). An experiment in linguistic synthesis with a fuzzy logic controller, *International Journal of Man-Machine Studies* **20**(2): 1–13.

Mazandarani, M. and Xiu, L. (2020). Fractional fuzzy inference system: The new generation of fuzzy inference systems, *IEEE Access* **8**: 126066–126082.

Mizumoto, M. and Zimmermann, H.-J. (1982). Comparison of fuzzy reasoning methods, *Fuzzy Sets and Systems* **8**(3): 253–283.

Piegat, A. and Dobryakova, L. (2020). A decomposition approach to type 2 interval arithmetic, *International Journal of Applied Mathematics and Computer Science* **30**(1): 185–201, DOI: 10.34768/amcs-2020-0015.

Rutkowski, L. (2008). *Computational Intelligence, Methods and Techniques*, Springer, Berlin/Heidelberg.

Tong, R.M. and Festathiou, J. (1982). A critical assessment of truth function modification and its use in approximate reasoning, *Fuzzy Sets and Systems* **7**(1): 103–108.

Ughetto, L., Dubois, D. and Prade, H. (1999). Implicative and conjunctive fuzzy rules—A tool for reasoning from knowledge and examples, *16th National Conference on Artificial Intelligence/11th Annual Conference on Innovative Applications of Artificial Intelligence, Orlando, USA*, pp. 214–219.

Yagger, R. (1996). On the interpretation of fuzzy if-then rules, *Applied Intelligence* **6**(2): 141–151.

Zadeh, L. (1973). Outline of a new approach to the analysis of complex systems and decision processes, *IEEE Transactions on Systems, Man and Cybernetics* **3**(1): 28–44.

Zadeh, L. (1975). Fuzzy logic and approximate reasoning, *Syntheses* **30**(3): 407–428.

Zimmermann, H.-J. (1985). *Fuzzy Set Theory and Its Applications*, Springer, Dordrecht.

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