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## Linear functional equations and their solutions in generalized Orlicz spaces

JANUSZ MORAWIEC  AND THOMAS ZÜRCHER

*Dedicated to Professor Ludwig Reich on the occasion of his 80th birthday.*

**Abstract.** Assume that  $\Omega \subset \mathbb{R}^k$  is an open set,  $V$  is a real separable Banach space and  $f_1, \dots, f_N: \Omega \rightarrow \Omega$ ,  $g_1, \dots, g_N: \Omega \rightarrow \mathbb{R}$ ,  $h_0: \Omega \rightarrow V$  are given functions. We are interested in the existence and uniqueness of solutions  $\varphi: \Omega \rightarrow V$  of the linear equation  $\varphi = \sum_{k=1}^N g_k \cdot (\varphi \circ f_k) + h_0$  in generalized Orlicz spaces.

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### 1. Introduction

Throughout this paper we fix  $k, N \in \mathbb{N}$ , an open set  $\Omega \subset \mathbb{R}^k$ , a real separable Banach space  $(V, \|\cdot\|_V)$  and functions  $f_1, \dots, f_N: \Omega \rightarrow \Omega$ ,  $g_1, \dots, g_N: \Omega \rightarrow \mathbb{R}$  and  $h_0: \Omega \rightarrow V$ . We are interested in solutions  $\varphi: \Omega \rightarrow V$  of the linear equation of the form

$$\varphi(x) = \sum_{n=1}^N g_n(x) \varphi(f_n(x)) + h_0(x). \quad (1)$$

Solutions of Eq. (1) have been studied by many authors in different classes of functions (for more details see e.g. [12, Chapter XIII], [13, Chapter 6], [2, Chapter 5], [1, Section 4] and the references therein). In this paper we are interested in the existence and uniqueness of solutions of Eq. (1) in generalized Orlicz spaces.

This paper is a continuation of investigations started by the authors in [17], where solutions of Eq. (1) were studied in the space  $L^1([0, 1], \mathbb{R})$ . The

motivation to study Eq. (1) in the space  $L^1([0, 1], \mathbb{R})$  came from [20]. However, the interest to consider it in much more general spaces is inspired by [15].

We denote by  $\mathcal{F}$  the linear space of all functions  $\psi: \Omega \rightarrow V$  and fix a subspace  $\mathcal{F}_0$  of  $\mathcal{F}$ . Then we define the operator  $P: \mathcal{F}_0 \rightarrow \mathcal{F}$  by

$$P\psi = \sum_{n=1}^N g_n \cdot (\psi \circ f_n), \quad (2)$$

and we observe that it is linear and Eq. (1) can be written in the form

$$\varphi = P\varphi + h_0. \quad (3)$$

Note also that if Eq. (1) has a solution  $\varphi \in \mathcal{F}_0$  such that  $P\varphi \in \mathcal{F}_0$ , then  $h_0 \in \mathcal{F}_0$ . Conversely, if  $h_0 \in \mathcal{F}_0$ , then for every solution  $\varphi \in \mathcal{F}_0$  of Eq. (1) we have  $P\varphi \in \mathcal{F}_0$ . Therefore, if we want to look for solutions of Eq. (1) in  $\mathcal{F}_0$ , then it is quite natural to assume that  $h_0 \in \mathcal{F}_0$  and

$$P(\mathcal{F}_0) \subset \mathcal{F}_0. \quad (4)$$

We begin with the following counterpart of [17, Remark 1.2].

*Remark 1.1.* Assume that  $\mathcal{F}_0$  is equipped with a norm,  $h_0 \in \mathcal{F}_0$  and the operator  $P$  given by (2) satisfies (4) and is continuous. If the series

$$\sum_{n=0}^{\infty} P^n h_0 \quad (5)$$

converges, in the norm, to a function  $\varphi \in \mathcal{F}_0$ , then (3) holds.

From now on, the series (5) will be called the *elementary solution* of Eq. (1) in  $\mathcal{F}_0$ , provided that it is a well-defined solution of Eq. (1) belonging to  $\mathcal{F}_0$ . Let us note that it can happen that Eq. (1) has a solution in  $\mathcal{F}_0$ , however its elementary solution in  $\mathcal{F}_0$  can fail to exist (see [17, Example 1.4]). Following [17] we are interested in assumptions guaranteeing that the elementary solution of Eq. (1) in  $\mathcal{F}_0$  exists, and moreover, that Eq. (1) has no other solutions in  $\mathcal{F}_0$ . As it was mentioned at the beginning, in this paper we will focus on  $\mathcal{F}_0$  when it is a generalized Orlicz space.

Our first result is a simple generalization of [17, Theorem 3.2], essentially with the same proof.

**Theorem 1.2.** *Assume that  $\|\cdot\|$  is a complete norm in  $\mathcal{F}_0$  and let  $h_0 \in \mathcal{F}_0$ . If the operator  $P$  given by (2) satisfies (4) and is a contraction with contraction factor  $\alpha$ , then the elementary solution of Eq. (1) in  $\mathcal{F}_0$  exists, it is the unique solution of Eq. (1) in  $\mathcal{F}_0$  and  $\|\sum_{k=m}^{\infty} P^k h_0\| \leq \frac{\alpha^m}{1-\alpha} \|h_0\|$ .*

## 2. Preliminaries

Let  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  be measure spaces. We say that  $G: X \rightarrow Y$  satisfies *Luzin's condition N* if for every set  $N \subset Y$  of measure zero the set  $G(N)$  is also of measure zero. When we integrate a function  $\Phi: X \rightarrow V$ , we will use the Bochner integral (for details see e.g. [8, Sections 3.1 and 3.2]). Recall that a function  $\Phi: X \rightarrow V$  is *Bochner-measurable* if it is equal almost everywhere to the limit of a sequence of measurable simple functions, i.e.,  $\Phi(x) = \lim_{n \rightarrow \infty} \Phi_n(x)$  for almost all  $x \in X$ , where each of the functions  $\Phi_n: X \rightarrow V$  has a finite range and  $\Phi_n^{-1}(\{v\})$  is measurable for every  $v \in V$ . As we will work with Bochner-integrable solutions of Eq. (1), we need the following observation.

**Lemma 2.1.** *Assume that  $(X, \mathcal{M}, \mu)$  is a complete  $\sigma$ -finite measure space. Let  $F: X \rightarrow X$ ,  $H: X \rightarrow V$  and  $G: X \rightarrow \mathbb{R}$  be measurable functions. If for all sets  $N \subset X$  of measure zero the set  $F^{-1}(N)$  is also of measure zero, then the functions  $H \circ F$  and  $G \cdot (H \circ F)$  are measurable.*

*Proof.* Let  $(H_n)_{n \in \mathbb{N}}$  be a sequence of measurable simple functions converging pointwise to  $H$  except on a set  $N$  of measure zero. Fix  $n \in \mathbb{N}$ . Then there are points  $v_1, \dots, v_m \in V$  and a partition of  $V$  into measurable sets  $V_1, \dots, V_m$  such that  $H_n = \sum_{j=1}^m v_j \chi_{V_j}$ . Then  $H_n \circ F = \sum_{j=1}^m v_j \chi_{V_j} \circ F = \sum_{j=1}^m v_j \chi_{F^{-1}(V_j)}$ . By our assumptions, the set  $F^{-1}(N)$  is of measure zero. Fix  $v \in V \setminus F^{-1}(N)$ . Then  $F(v) \notin N$ , and hence  $(H_n \circ F)(v)$  converges to  $(H \circ F)(v)$ .

Assume now that  $(G_n)_{n \in \mathbb{N}}$  is a sequence of measurable simple functions converging to  $G$  except on a set  $M$  of measure zero. Then  $(G_n \cdot (H_n \circ F))_{n \in \mathbb{N}}$  is a sequence of measurable simple functions converging to  $G \cdot (H \circ F)$  except on the set  $F^{-1}(N) \cup M$ .  $\square$

The next result we want to apply is a change of variable formula from [6]. To formulate this theorem, we need to introduce some definitions and notions.

Let  $F: \Omega \rightarrow \mathbb{R}^k$  be measurable. We say that a linear mapping  $L: \mathbb{R}^k \rightarrow \mathbb{R}^k$  is an *approximate differential* of  $F$  at  $x_0 \in \Omega$  if for every  $\varepsilon > 0$  the set

$$\left\{ x \in \Omega \setminus \{x_0\} : \frac{\|F(x) - F(x_0) - L(x - x_0)\|}{\|x - x_0\|} < \varepsilon \right\}$$

has  $x_0$  as a density point (see [24, Section 2], cf. [23, Chapter IX.12]). We say that  $F$  is *approximately differentiable* at  $x_0$  if the approximate differential of  $F$  at  $x_0$  exists. To simplify notation, we will denote the approximate differential of a function  $F: \Omega \rightarrow \mathbb{R}^k$  at  $x_0$  by  $F'(x_0)$ . Moreover, if a function  $F: \Omega \rightarrow \mathbb{R}^k$  is almost everywhere approximately differentiable, then as usual we denote by  $F'$  the function  $\Omega \ni x \mapsto F'(x)$ , adopting the convention that  $F'(x) = 0$  for every point  $x \in \Omega$  at which  $F$  is not approximately differentiable. If  $E \subset \Omega$ , then the function  $N_F(\cdot, E): \mathbb{R}^k \rightarrow \mathbb{N} \cup \{\infty\}$  defined by

$$N_F(y, E) = \text{card}(F^{-1}(y) \cap E)$$

is called the *Banach indicatrix* of  $F$ .

We omit the proof of the next lemma, as it is the same as the version included in [17] in the case where  $k = 1$ .

**Lemma 2.2.** (see [17, Lemma 2.1]) *Assume that  $F_1, F_2: \Omega \rightarrow \mathbb{R}^k$  are functions such that  $F_1 = F_2$  almost everywhere. If  $F_1$  is approximately differentiable almost everywhere, then so is  $F_2$ . Moreover, whenever  $F_1$  or  $F_2$  is approximately differentiable at a point, so is the other function, and the approximate derivatives agree at this point.*

Now we are in a position to formulate the change of variable formula;  $J_F$  denotes the Jacobian of  $F$ .

**Theorem 2.3.** (see [6, Theorem 2]) *Assume that  $F: \Omega \rightarrow \mathbb{R}^k$  is a measurable function satisfying Luzin's condition  $N$  and is almost everywhere approximately differentiable. If  $H: \mathbb{R}^k \rightarrow \mathbb{R}$  is a measurable function, then for every measurable set  $E \subset \Omega$  the following statements are true:*

- (i) *The functions  $(H \circ F)|J_F|$  and  $HN_F(\cdot, E)$  are measurable;*
- (ii) *If  $H \geq 0$ , then*

$$\int_E (H \circ F)(x)|J_F(x)| dx = \int_{\mathbb{R}^k} H(y)N_F(y, E) dy; \tag{6}$$

- (iii) *If one of the functions  $(H \circ F)|J_F|$  and  $HN_F(\cdot, E)$  is integrable (for  $(H \circ F)|J_F|$  integrability is considered with respect to  $E$ ), then so is the other and (6) holds.*

Now we are ready to formulate the main assumption about the functions that were fixed at the beginning of this paper. The assumption reads as follows.

- (H) *The functions  $f_1, \dots, f_N$  are measurable and almost everywhere approximately differentiable and satisfy Luzin's condition  $N$ . For all  $n \in \{1, \dots, N\}$  and sets  $M \subset \mathbb{R}^k$  of measure zero the set  $f_n^{-1}(M)$  is of measure zero. There exists  $K \in \mathbb{N}$  such that for every  $n \in \{1, \dots, N\}$  the set  $\{x \in \Omega : \text{card } f_n^{-1}(x) > K\}$  is of measure zero. The functions  $g_1, \dots, g_N$  and  $h_0$  are measurable.*

From now on, for all  $l \in \{1, \dots, N\}$  and distinct  $n_1, \dots, n_l \in \{1, \dots, N\}$  we put

$$A_{n_1, \dots, n_l} = \bigcap_{i=1}^l f_{n_i}(\Omega)$$

and denote its measure by  $l_{n_1, \dots, n_l}$ . Then we set

$$L = \max\{l \in \{1, \dots, N\} : l_{n_1, \dots, n_l} > 0 \text{ for some } n_1 < n_2 < \dots < n_l\}.$$

### 3. Lebesgue spaces

We begin our considerations on the existence and uniqueness of the elementary solution of Eq. (1) in Lebesgue spaces  $L^p(\Omega, V)$  of vector-valued functions. As the Lebesgue spaces of vector-valued functions are well known as natural generalizations of the classical Lebesgue spaces of real-valued functions (see e.g. [3] or [8, Chapter 3]), we will not define and describe them in details here.

Since we want to apply Theorem 1.2, we must know that the operator  $P$  given by (2) has the properties assumed in this theorem. The next two lemmas serve this purpose.

**Lemma 3.1.** *Assume that (H) holds and let  $p \in [1, \infty)$ . If there exists a real constant  $\alpha \geq 0$  such that*

$$|g_n(x)|^p \leq \frac{\alpha^p}{KN^p} |J_{f_n}(x)| \quad \text{for all } n \in \{1, \dots, N\} \text{ and almost all } x \in \Omega, \quad (7)$$

*then the operator  $P$  given by (2) satisfies  $P(L^p(\Omega, V)) \subset L^p(\Omega, V)$  and is continuous with*

$$\|P\| \leq \alpha. \quad (8)$$

*Proof.* The proof is similar to that of [17, Lemma 3.1].

Fix  $h \in L^p(\Omega, V)$ . First of all observe that applying assertion (i) of Theorem 2.3 with  $F = f_n$  and  $H = 1$  we conclude that the function  $N_{f_n}(\cdot, \Omega)$  is measurable for every  $n \in \{1, \dots, N\}$ . Hence the function  $\|h(\cdot)\|_V N_{f_n}(\cdot, \Omega)$  is also measurable for every  $n \in \{1, \dots, N\}$ . Next by Lemma 2.1 we see that the function  $g_n \cdot (h \circ f_n)$  is measurable for every  $n \in \{1, \dots, N\}$ , which implies that the function  $Ph$  is measurable as well. Then, using (7) and Theorem 2.3, we obtain

$$\begin{aligned} \|g_n \cdot (h \circ f_n)\|_{L^p(\Omega, V)}^p &\leq \int_{\Omega} |g_n(x)|^p \|(h \circ f_n)(x)\|_V^p dx \\ &\leq \frac{\alpha^p}{N^p} \frac{1}{K} \int_{\Omega} \|(h \circ f_n)(x)\|_V^p |J_{f_n}(x)| dx \\ &= \frac{\alpha^p}{N^p} \frac{1}{K} \int_{f_n(\Omega)} \|h(y)\|_V^p N_{f_n}(y, \Omega) dy \\ &\leq \frac{\alpha^p}{N^p} \int_{f_n(\Omega)} \|h(y)\|_V^p dy \leq \frac{\alpha^p}{N^p} \|h\|_V^p. \end{aligned}$$

This yields

$$\|Ph\|_{L^p(\Omega, V)} \leq \sum_{n=1}^N \|g_n \cdot (h \circ f_n)\|_{L^p(\Omega, V)} \leq \alpha \|h\|_{L^p(\Omega, V)}$$

and completes the proof.  $\square$

**Lemma 3.2.** *Assume that (H) holds. If*

$$\alpha = \sum_{n=1}^N \|g_n\|_{L^\infty(\Omega, \mathbb{R})} < \infty, \quad (9)$$

*then the operator  $P$  given by (2) satisfies  $P(L^\infty(\Omega, V)) \subset L^\infty(\Omega, V)$ , it is continuous and (8) holds.*

*Proof.* It is enough to observe that

$$\|Ph\|_{L^\infty(\Omega, V)} \leq \sum_{n=1}^N \|g_n(h \circ f_n)\|_{L^\infty(\Omega, V)} \leq \alpha \|h\|_{L^\infty(\Omega, V)}$$

for every  $h \in L^\infty(\Omega, V)$ . □

It is well known that Lebesgue spaces of real-valued functions are Banach spaces (see e.g. [5, Theorem 6.6]). It turns out that the same is true in the case of vector-valued functions with Banach target spaces, basically with the same proof as in the real-valued case (see e.g. [8, Section 3.2]). Therefore, applying Theorem 1.2 jointly with Lemmas 3.1 and 3.2 we obtain the following result.

**Theorem 3.3.** *Assume that (H) holds, that  $p \in [1, \infty]$  and that  $h_0 \in L^p(\Omega, V)$ . Let (7) be satisfied with some  $\alpha \geq 0$  in the case where  $p \in [1, \infty)$  and (9) be satisfied in the case where  $p = \infty$ . If  $\alpha < 1$ , then the elementary solution of Eq. (1) in  $L^p(\Omega, V)$  exists, it is the unique solution of Eq. (1) in  $L^p(\Omega, V)$  and*

$$\left\| \sum_{k=m}^{\infty} P^k h_0 \right\|_{L^p(\Omega, V)} \leq \frac{\alpha^m}{1 - \alpha} \|h_0\|_{L^p(\Omega, V)}.$$

The next results concerns the space  $C(F, V)$  of all continuous functions from a compact set  $F \subset \mathbb{R}^k$  to  $V$  equipped with the supremum norm  $\|\cdot\|_{\text{sup}}$ . Although it is not a Lebesgue space, we will formulate a counterpart of Theorem 3.3 for it. The reason is that its proof is based on the following lemma, the proof of which is the same as the proof of Lemma 3.2.

**Lemma 3.4.** *Assume (H) and let  $F \subset \Omega$  be a non-empty compact set. If for every  $n \in \{1, \dots, N\}$  the functions  $f_n$  and  $g_n$  are continuous on  $F$  with  $f_n(F) \subset F$ , and if*

$$\alpha = \sum_{n=1}^N \|g_n\|_{\text{sup}} < \infty \quad (10)$$

*then the operator  $P$  given by (2) satisfies  $P(C(F, V)) \subset C(F, V)$ , is continuous and (8) holds.*

Again, since  $C(F, V)$  with the supremum norm is a Banach space (see e.g. [3, Introduction]), it follows that Theorem 1.2 jointly with Lemma 3.4 gives the following result.

**Theorem 3.5.** *Assume that (H) holds and let  $F \subset \Omega$  be a non-empty compact set, let  $h_0$  be continuous on  $F$ , let for every  $n \in \{1, \dots, N\}$  the functions  $f_n$  and  $g_n$  be continuous on  $F$  with  $f_n(F) \subset F$ , and let (10) be satisfied. If  $\alpha < 1$ , then the elementary solution of Eq. (1) in  $C(F, V)$  exists, it is the unique solution of Eq. (1) in  $C(F, V)$  and*

$$\left\| \sum_{k=m}^{\infty} P^k h_0 \right\|_{\sup} \leq \frac{\alpha^m}{1 - \alpha} \|h_0\|_{\sup}.$$

#### 4. Generalized Orlicz spaces

In this section we will focus on generalized Orlicz spaces (called also Musielak–Orlicz spaces) with values in Banach spaces. Such spaces are a known generalization of the classical Orlicz spaces, and hence they are more general than Lebesgue spaces. Generalized Orlicz spaces were introduced in the case of real valued function in [18] and then generalized also to functions taking values in vector spaces in [10]. There are many results obtained on generalized Orlicz spaces (see e.g. [7, 16, 19, 25] and the references therein). For the convenience of the readers, following [10, 11], we recall some basic definitions and facts for our needs.

Denote by  $\mathcal{B}(V)$  the  $\sigma$ -algebra of all Borel subsets of  $V$  and by  $\mathcal{L}_k(\Omega)$  the  $\sigma$ -algebra of all Lebesgue measurable subsets of  $\Omega$ .

**Definition 4.1.** [see [10, Definitions 2.1.1, 2.1.2 and 2.1.3]; cf. [11, Section 0]] A function  $\Phi: V \times \Omega \rightarrow [0, \infty]$  is said to be an  $N$ -function if:

- (i)  $\Phi$  is  $\mathcal{B}(V) \otimes \mathcal{L}_k(\Omega)$ -measurable,
- (ii)  $\Phi(\cdot, x)$  is even, convex, continuous at zero and lower semicontinuous for almost all  $x \in \Omega$ ,
- (iii)  $\Phi(0, x) = 0$  for almost all  $x \in \Omega$ ,
- (iv) there exist functions  $\alpha, \beta: \Omega \rightarrow (0, \infty)$  such that  $\Phi(v, x) \geq \alpha(x)$  for all  $v \in V$  and almost all  $x \in \Omega$  with  $\|v\|_V \geq \beta(x)$ .

From now on the symbol  $\Phi$  is reserved for  $N$ -functions only.

Denote by  $\mathcal{M}_V$  the set of all measurable functions  $h: \Omega \rightarrow V$ ; as usual, two functions from  $\mathcal{M}_V$  that differ only on a set of measure zero will be considered as equal. Assume that  $\mathcal{M}$  is a given non-empty subset of  $\mathcal{M}_V$  such that

$$I_{\Phi}(h) = \int_{\Omega} \Phi(h(x), x) dx < \infty \quad \text{for every } h \in \mathcal{M}.$$

Denote by  $L_{\mathcal{M}}^{\Phi}(\Omega, V)$  the set of all functions  $h \in \mathcal{M}$  for which there exists a sequence  $(h_n)_{n \in \mathbb{N}}$  of functions belonging to  $\text{lin } \mathcal{M}$ , i.e. the smallest linear space spanned by  $\mathcal{M}$ , such that

$$\lim_{n \rightarrow \infty} I_{\Phi}(\xi(h_n - h)) = 0 \quad \text{for every } \xi > 0. \tag{11}$$



It turns out that  $L_{\mathcal{M}}^{\Phi}(\Omega, V)$  is a linear space; indeed it is enough to note that  $I_{\Phi}(\xi(\alpha h_n - \tilde{h}_n - (\alpha h - \tilde{h}))) \leq \frac{1}{2}I_{\Phi}(2|\alpha|\xi(h_n - h)) + \frac{1}{2}I_{\Phi}(2\xi(\tilde{h}_n - \tilde{h}))$  for all  $h, \tilde{h}, h_n, \tilde{h}_n \in \mathcal{M}_V$ ,  $\alpha \in \mathbb{R}$  and  $\xi > 0$ . The linear space  $L_{\mathcal{M}}^{\Phi}(\Omega, V)$  is called a *generalized Orlicz space* (or a *Musielałak–Orlicz space*).

Typical generalized Orlicz spaces are variable exponent Lebesgue spaces  $L^{p(\cdot)}(\Omega, V)$ , generated by  $N$ -functions of the form  $\Phi(v, x) = \|v\|_V^{p(x)}$  with measurable functions  $p: \Omega \rightarrow [1, \infty)$ , and double phase spaces (containing  $L^p(\Omega, V) + L^q(\Omega, V)$  spaces), generated by  $N$ -functions of the form  $\Phi(v, x) = \|v\|_V^p + \alpha(x)\|v\|_V^q$  with measurable functions  $\alpha: \Omega \rightarrow (0, \infty)$  and real numbers  $p, q \in [1, \infty)$ . Additional interesting examples of generalized Orlicz spaces can be found in [7].

**Theorem 4.2.** [see [10, Theorem 2.4]] *The formula*

$$\|h\|_{L_{\mathcal{M}}^{\Phi}(\Omega, V)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi\left(\frac{h(x)}{\lambda}, x\right) dx \leq 1 \right\}$$

*defines a complete norm in  $L_{\mathcal{M}}^{\Phi}(\Omega, V)$ .*

Let us note that in the above theorem the assumption about the continuity at zero of  $\Phi(\cdot, x)$  in condition (ii) of Definition 4.1 can be omitted. However, we will need this assumption to simplify our results. For the same reason we will work only with the generalized Orlicz spaces  $L_{\mathcal{M}_V}^{\Phi}(\Omega, V)$ , denoted throughout this paper by  $L^{\Phi}(\Omega, V)$ . The norm introduced in Theorem 4.2 is called the *Luxemburg norm*.

From [10, Proposition 2.2] it follows that  $L^{\Phi}(\Omega, V) = \text{lin } \mathcal{M}$ , which correlates the generalized Orlicz space with the original one introduced in [21]. Moreover, [10, Proposition 2.3] yields  $I_{\Phi}(h) < \infty$  for every  $h \in L^{\Phi}(\Omega, V)$ , which jointly with the continuity at zero of the function  $\Phi(\cdot, x)$  leads to the following observation.

**Theorem 4.3.** [see [22, Theorem 1.13]] *We have  $h \in L^{\Phi}(\Omega, V)$  if and only if one of the following two equivalent conditions hold:*

- (i) *there exists a sequence  $(h_n)_{n \in \mathbb{N}}$  of functions from  $\text{lin}\{\tilde{h} \in \mathcal{M}_V : I_{\Phi}(\tilde{h}) < \infty\}$  satisfying (11),*
- (ii) *there exists  $\xi > 0$  such that  $I_{\Phi}(\xi h) < \infty$ .*

After the short introduction to the generalized Orlicz space, we pass to our investigations.

We are now ready to generalize Lemma 3.1 to generalized Orlicz spaces.

**Lemma 4.4.** *Assume that (H) holds. If there exists a real constant  $\alpha \geq 0$  such that*

$$\Phi(P_h(x), x) \leq \frac{1}{KL} \sum_{n=1}^N \Phi(\alpha h(f_n(x)), f_n(x)) |J_{f_n}(x)| \tag{12}$$

*for all  $h \in L^{\Phi}(\Omega, V)$  and almost all  $x \in \Omega$ ,*

then the operator  $P$  given by (2) satisfies  $P(L^\Phi(\Omega, V)) \subset L^\Phi(\Omega, V)$ , it is continuous and (8) holds.

*Proof.* Fix  $h \in L^\Phi(\Omega, V)$ . As in the proof of Lemma 3.1 we conclude that the functions  $Ph$  and  $N_{f_n}(\cdot, \Omega)$  are measurable for every  $n \in \{1, \dots, N\}$ . According to [9, Theorem 2 in Appendix 2 on page 68] (see also [22, page 323]; cf. [7, Theorem 2.5.4]) we deduce that both functions  $\Phi(h(\cdot), \cdot)N_{f_n}(\cdot, \Omega)$  and  $\Phi(Ph(\cdot), \cdot)$  are also measurable for every  $n \in \{1, \dots, N\}$ . Then using (12) and Theorem 2.3, we obtain

$$\begin{aligned}
 \|Ph\|_{L^\Phi(\Omega, V)} &= \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi \left( \frac{Ph(x)}{\lambda}, x \right) dx \leq 1 \right\} \\
 &\leq \inf \left\{ \lambda > 0 : \frac{1}{KL} \sum_{n=1}^N \int_{\Omega} \Phi \left( \frac{\alpha}{\lambda} h(f_n(x)), f_n(x) \right) |J_{f_n}(x)| dx \leq 1 \right\} \\
 &= \inf \left\{ \lambda > 0 : \frac{1}{KL} \sum_{n=1}^N \int_{\Omega} \Phi \left( \frac{\alpha}{\lambda} h(y), y \right) N_{f_n}(y, \Omega) dy \leq 1 \right\} \\
 &\leq \inf \left\{ \lambda > 0 : \frac{1}{L} \sum_{n=1}^N \int_{f_n(\Omega)} \Phi \left( \frac{\alpha}{\lambda} h(y), y \right) dy \leq 1 \right\} \\
 &\leq \inf \left\{ \lambda > 0 : \int_{\bigcup_{n=1}^N f_n(\Omega)} \Phi \left( \frac{\alpha}{\lambda} h(y), y \right) dy \leq 1 \right\} \\
 &\leq \alpha \inf \left\{ \eta > 0 : \int_{\Omega} \Phi \left( \frac{h(y)}{\eta}, y \right) dy \leq 1 \right\} = \alpha \|h\|_{L^\Phi(\Omega, V)},
 \end{aligned}$$

which completes the proof.  $\square$

Note that in condition (12) all functions in the space  $L^\Phi(\Omega, V)$  are involved, which makes it a bit difficult to check. However, we can formulate a simple condition that involves no function of the space  $L^\Phi(\Omega, V)$  and implies (12).

*Remark 4.5.* If

$$\Phi(v, x) = \Phi(v, f_n(x)) \quad \text{for all } n \in \{1, \dots, N\} \text{ and almost all } x \in \Omega \quad (13)$$

and

$$\begin{aligned}
 |g_n(x)| \leq \alpha \min \left\{ \frac{|J_{f_n}(x)|}{KL}, \frac{1}{N} \right\} \quad &\text{for all } n \in \{1, \dots, N\} \text{ and} \\
 &\text{almost all } x \in \Omega,
 \end{aligned} \quad (14)$$

then (12) holds.

*Proof.* First observe that if  $\alpha = 0$ , then (14) gives  $Ph(x) = 0$  for almost all  $x \in \Omega$ , and hence (12) holds. Therefore, we can assume that  $\alpha > 0$ .

Fix  $x \in \Omega$  such that  $\Phi(0, x) = 0$  and  $\Phi(\cdot, x)$  is convex and even. Then for all  $v_1, v_2 \in V$  and  $a, b \in \mathbb{R}$  with  $|a| + |b| \leq 1$  we have

$$\begin{aligned} \Phi(av_1 + bv_2, x) &= \Phi(|a| \operatorname{sgn}(a)v_1 + |b| \operatorname{sgn}(b)v_2 + (1 - |a| - |b|)0, x) \\ &\leq |a|\Phi(\operatorname{sgn}(a)v_1, x) + |b|\Phi(\operatorname{sgn}(b)v_2, x) + (1 - |a| - |b|)\Phi(0, x) \\ &= |a|\Phi(v_1, x) + |b|\Phi(v_2, x). \end{aligned}$$

This jointly with (14), the non-negativity of  $\Phi$ , and (13) gives

$$\begin{aligned} \Phi(Ph(x), x) &= \Phi\left(\sum_{n=1}^N g_n(x)h(f_n(x)), x\right) \leq \frac{1}{\alpha} \sum_{n=1}^N |g_n(x)|\Phi(\alpha h(f_n(x)), x) \\ &\leq \frac{1}{KL} \sum_{n=1}^N \Phi(\alpha h(f_n(x)), x) |J_{f_n}(x)| \\ &= \frac{1}{KL} \sum_{n=1}^N \Phi(\alpha h(f_n(x)), f_n(x)) |J_{f_n}(x)| \end{aligned}$$

for almost all  $x \in \Omega$ . □

Combining Lemma 4.4 and Theorems 4.2 and 1.2, we obtain the following result.

**Theorem 4.6.** *Assume that (H) holds and let  $h_0 \in L^\Phi(\Omega, V)$ . If (12) holds with a real constant  $\alpha \in [0, 1)$ , then the elementary solution of Eq. (1) in  $L^\Phi(\Omega, V)$  exists, it is the unique solution of Eq. (1) in  $L^\Phi(\Omega, V)$  and*

$$\left\| \sum_{k=m}^{\infty} P^k h_0 \right\|_{L^\Phi(\Omega, V)} \leq \frac{\alpha^m}{1 - \alpha} \|h_0\|_{L^\Phi(\Omega, V)}.$$

We now introduce an interesting and widely studied class of  $N$ -functions.

**Definition 4.7.** (see [14, Definitions 2.2]) A non-decreasing left-continuous and convex function  $\Psi: [0, \infty) \rightarrow [0, \infty]$  is said to be a *Young function*, if

$$\lim_{t \rightarrow 0^+} \Psi(t) = \Psi(0) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \Psi(t) = \infty.$$

From now on the symbol  $\Psi$  is reserved for Young functions only.

It is easy to check that if  $\Psi$  is a strictly increasing Young function, then the formula

$$\Phi(v, x) = \Psi(\|v\|_V)$$

defines an  $N$ -function. Therefore, we see that every function  $\Psi$  produces an Orlicz space of the form

$$L^\Psi(\Omega, V) = \left\{ h \in \mathcal{M}_V : \int_{\Omega} \Psi(\xi \|h(x)\|_V) dx < \infty \text{ with some } \xi > 0 \right\}.$$

This space equipped with the Luxemburg norm

$$\|h\|_{L^\Psi(\Omega, V)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \Psi \left( \frac{\|h(x)\|_V}{\lambda} \right) dx \leq 1 \right\}$$

is complete by Theorem 4.2 (cf. [14, Definition 2.7]). Note that in general the set  $S = \{h \in \mathcal{M}_V : \int_{\Omega} \Psi(\|h(x)\|_V) dx < \infty\}$  is not a vector space (for an easy example see e.g. [4, page 96]), however assuming an additional property on  $\Psi$  the set  $S$  becomes a vector space (see e.g. [4, Theorem 3.4.13]).

Summarizing, we formulate the following result for the Orlicz spaces  $L^\Psi(\Omega, V)$ .

**Corollary 4.8.** *Assume that (H) holds and let  $h_0 \in L^\Psi(\Omega, V)$ . If (14) is satisfied with a real constant  $\alpha \in [0, 1)$ , then the elementary solution of Eq. (1) in  $L^\Psi(\Omega, V)$  exists, it is the unique solution of Eq. (1) in  $L^\Psi(\Omega, V)$  and*

$$\left\| \sum_{k=m}^{\infty} P^k h_0 \right\|_{L^\Psi(\Omega, V)} \leq \frac{\alpha^m}{1-\alpha} \|h_0\|_{L^\Psi(\Omega, V)}.$$

*Proof.* It is enough to apply Theorem 4.6 and Remark 4.5 noting that Eq. 13 holds as  $\Psi$  does not depend on  $x$ . □

Finally note that since for every  $p \in (1, \infty)$  the mapping “ $[0, \infty) \ni t \mapsto t^p$ ” is a Young function, it follows that Corollary 4.8 extends Theorem 3.3 in the case where  $p \in (1, \infty)$ ; cf. [4, Remark 3.4.18].

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