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Symmetry of Syzygies of a System of Functional Equations Defining a Ring Homomorphism

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Abstract: I deal with an alienation problem for the system of two fundamental Cauchy functional equations with an unknown function f mapping a ring X into an integral domain Y and preserving binary operations of addition and multiplication, respectively. The resulting syzygies obtained by adding (resp. multiplying) these two equations side by side are discussed. The first of these two syzygies was first examined by Jean Dhombres in 1988 who proved that under some additional conditions concering the domain and range rings it forces f to be a ring homomorphism (alienation phenomenon). The novelty of the present paper is to look for sufficient conditions upon f solving the other syzygy to be alien.

Keywords: functional equations; alienation; homomorphism between rings; additivity; multiplicativity

MSC: 39B52; 39B72; 39B82

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1. Introduction

It seems hardly likely that given a map f between two rings X and Y, the corresponding *syzygies*

$$f(x+y)+f(xy)=f(x)+f(y)+f(x)f(y) \tag{1}$$

or

$$f(x+y)f(xy)=(f(x)+f(y))f(x)f(y) \tag{2}$$

of the system

$$\begin{cases} f(x+y) = f(x)+f(y) \\ f(xy) = f(x)f(y) \end{cases}$$

of two Cauchy functional equations defining a homomorphism of the rings X and Y might bring us back to the additivity of f and hence also the multiplicativity of f (alienation phenomenon). Nevertheless, bearing in mind the results obtained by J. Dhombres [1] (where the alienation idea comes from), and, as an example, the papers of W. Fechner [2], R. Ger and L. Reich [3], Z. Kominek and J. Sikorska [4], Gy. Maksa and M. Sablik [5] and B. Sobek [6], such a conjecture regarding Equation (2) becomes more reasonable as well.

Moreover, four years ago, jointly with Maciej Sablik, I published a survey article [7] on the alienation phenomenon in the theory of functional equations. The reference list therein contains dozens of items concerning alienation.

The aim of the present paper is to visualize the perfect *symmetry* of the behavior of solutions of the syzygies (1) and (2).

To proceed, in what follows, we shall assume that X is a unitary ring with a unit $e \neq 0$ and Y is an integral domain (a commutative unitary ring (with a unit 1) with no zero divisors). The symbol $\widetilde{Y}$ will stand for the field of fractions of the ring Y. The same general assumptions were taken in the paper [3] of R. Ger and L. Reich, for instance.

2. Solutions with One or Two Element Ranges or with Large Kernels

Similarly as in the case of syzygy (1) (see, e.g., [3]), in general, the alienation phenomenon regarding (2) fails to hold even in the case of relatively simple rings as $\mathbb{Z}$ (integers) and $\mathbb{R}$ (the reals). For example, a straightforward verification proves that a constant function $f(x) \equiv \frac{1}{2}$ as well as function $f : \mathbb{Z} \longrightarrow \mathbb{R}$ given by the formula

$$f(x) = \begin{cases} 0 & for\ x \in 2\mathbb{Z}+1 \\ \frac{1}{2} & for\ x \in 2\mathbb{Z} \end{cases}$$

yield nonhomomorphic solutions to (2). As we shall see, in general, nonhomomorphic solutions may occur only in the case where f is constant or $\operatorname{card} f^{-1}(\{0\}) \geq 2$.

To simplify further statements, let us introduce a notion of a large subset of a given ring. Namely, we shall say that a subset A of the ring X is *large* provided that there exists an affine map $\varphi : X \longrightarrow X$ such that $A \cup \varphi(A) = X$; here, $\varphi(t) = at + b$, $t \in X$, with given coefficients a, b from X, $a \neq 0$.

Proposition 1. *Let $f : X \to Y$ be a solution to the equation*

$$f(x+y)f(xy) = (f(x) + f(y))f(x)f(y)\,,$$

and let

$$\ker f := \{x \in X : \ f(x) = 0\}.$$

Then, either $\operatorname{card} f(X) \leq 2$, or $\ker f = \{0\}$, or $\operatorname{card} \ker f \geq 2$. *In the latter case, for every nonzero element from* $\ker f$, *the following inclusion holds true:*

$$yX \subset \ker f \cup (y \ker f - y^2).$$

In particular, if either $e \in$ $\ker f$ *or the ring X is semisimple, then* $\ker f$ *is large; more precisely, for every nonzero element from* $\ker f$, *one has*

$$\ker f \cup (\ker f - e) = X \quad or \quad \ker f \cup \left(\frac{1}{y}\ker f + y\right) = X \quad or \quad \ker f \cup (y\ker f - y^2) = X,$$

depending on whether y is invertible or not, respectively.

Proof. Put $c := f(0)$ and set $y = 0$ in (2). Then, for every $x \in X$, one has

$$cf(x) = cf(x)(f(x) + c),$$

which, in the case where $c \neq 0$, implies that $f(x) = f(x)^2 + cf(x)$ for all $x \in X$ and, consequently, f has at most two values: 0 or $1 - c$.

From now on, we shall assume that $0 \in \ker f =: Z$ and $Z^* := Z \setminus \{0\} \neq \emptyset$. Obviously, for every $y \in Z^*$, the equality

$$f(x+y)f(xy) = 0 \tag{3}$$

is valid for all $x \in X$. Let us consider two cases:

(a) $e \in Z$
(b) $e \notin Z$.

Having (a), on setting $y = e$ in (3), we infer that, for every $x \in X$, one has $f(x+e)f(x) = 0$, whence $Z \cup (Z - e) = X$, i.e., Z is large with $\varphi(t) := t - e$, $t \in X$.

Having (b), fix arbitrarily a $y \in Z^*$. Note that, on account of (3), we deduce that, for every $x \in X$, one has $x + y \in Z$ or $xy \in Z$. Due to the symmetry of the right-hand side of (2), we obtain also the equality $f(x+y)[f(xy) - f(yx)] = 0$, whence $x + y \in Z$ or $yx \in Z$. Thus, in the case where y were invertible in X, the union $(Z - y) \cup \frac{1}{y}Z$ would coincide with the whole ring X, i.e., $Z \cup \left(\frac{1}{y}Z + y\right) = X$, stating that Z is large with $\varphi(t) := \frac{1}{y}t + y$, $t \in X$. If none of the members of Z^* were invertible, then the bilateral ideal $I_y := yX$ would be contained in the union $Z \cup (yZ - y^2)$.

Now, it is an easy task to prove the latter part of the assertion. $\square$

3. The Most Interesting Case: $\ker f = \{0\}$

From now on, we deal with solutions $f : X \to Y$ of Equation (2) with the property $f^{-1}(\{0\}) = \{0\}$.

Proposition 2. *Any such solution f enjoys the following properties:*

(i) *oddness;*
(ii) $f(e) = 1$;
(iii) $f(x+e) = f(x) + 1$, *for every* $x \in X$;
(iv) $f(x+2e) = f(x) + 2$, *for every* $x \in X$;
(v) $f(2e) = 2$;
(vi) $f(2x) = 2f(x)$, *for every* $x \in X$;
(vii) $f(x^2) = f(x)^2$, *for every* $x \in X$;
(viii) $f(x+2y) + f(2x+y) = 3f(x+y)$, *for all* $x, y \in X$;
(ix) $f(3x) = 3f(x)$, *for every* $x \in X$.

Proof. Ad (i). Fix arbitrarily an $x \in X \setminus \{0\}$ and put $y = -x$ in (2) to obtain the equality

$$0 = (f(x) + f(-x))f(x)f(-x)$$

and, consequently, the relation $f(-x) = -f(x)$, which is valid for $x = 0$ as well, because of the equality $f(0) = 0$.

Ad (ii). With $y = e$, Equation (2) states that

$$f(x+e)f(x) = f(x)^2 f(e) + f(x)f(e)^2 \quad \text{for all } x \in X.$$

In particular,

$$f(x+e) = f(x)f(e) + f(e)^2 \quad \text{for all } x \in X \setminus \{0\}.$$

Hence, on putting here $x = e$, we infer that

$$f(2e) = 2f(e)^2. \tag{4}$$

Now, setting $y = 2e$ in (2), we obtain

$$f(x+2e)f(2x) = f(x)^2 f(2e) + f(x)f(2e)^2 = f(x)^2 2f(e)^2 + f(x)4f(e)^4 = 2f(x)^2 f(e)^2 + 4f(x)f(e)^4 \tag{5}$$

for every $x \in X$, which, with $x = -e$, due to the oddness of f, gives

$$-f(e)f(2e) = f(e)f(-2e) = 2f(-e)^2 f(e)^2 + 4f(-e)f(e)^4 = 2f(e)^4 - 4f(e)^5.$$

Applying (4), we deduce that

$$-2f(e)^3 = 2f(e)^4 - 4f(e)^5 \quad \text{which implies the equality} \quad 2f(e)^2 - f(e) - 1 = 0.$$

Thus, $f(e) \in \{1, -\frac{1}{2}\} \subset \widetilde{Y}$. The possibility $f(e) = -\frac{1}{2}$ is excluded because, otherwise, by means of (4), we would have $f(2e) = \frac{1}{2}$ and, consequently, on account of (5),

$$f(x+2e)f(2x) = \frac{1}{2}f(x)^2 + \frac{1}{4}f(x) \quad \text{for all } x \in X,$$

which with $x = e$ implies that $f(3e)f(2e) = \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{4} \cdot \frac{1}{2} = 0$, a contradiction.

Ad (iii). Put $y = e$ in (2) to obtain, by means of (ii),

$$f(x+e)f(x) = f(x)^2 f(e) + f(x)f(e)^2 = f(x)^2 + f(x), \; x \in X,$$

whence $f(x+e) = f(x) + 1$ provided that $x \neq 0$. However, equality (ii) guarantees that the assertion in question is valid for $x = 0$ as well.

Ad (iv). Put $x+e$ in place of x in (iii) to obtain

$$f(x+2e) = f(x+e+e) = f(x+e)+1 = f(x)+1+1 = f(x)+2,\ x \in X.$$

Ad (v). Put $x=0$ in (iv).

Ad (vi) At first, we shall prove that (vi) holds true for all $x \in X$ such that $f(x) \neq -2$. In fact, applying (iv) and setting $y = 2e$ in (2), we arrive at

$$(f(x)+2)f(2x) = f(x+2e)f(2x) = f(x)^2 f(2e) + f(x)f(2e)^2 = 2f(x)^2 + 4f(x) = 2f(x)(f(x)+2)$$

for all $x \in X$ because of (v).

To finish the proof of property (vi), assume that $f(x_0) = -2$ for some $x_0 \in X$. Then, due to the oddness of f, we have $f(-x_0) = 2$, and setting $x = -x_0$ in (2), we obtain the equalities

$$f(y-x_0)f(-x_0 y) = f(-x_0)^2 f(y) + f(-x_0)f(y)^2 = 4f(y) + 2f(y)^2$$

valid for all y from X. In particular, for $y = -x_0$, we obtain

$$-4f(2x_0) = 4f(-2x_0) = f(-2x_0)f(x_0)^2 = 4f(-x_0) + 2f(-x_0)^2 = 16 = -8f(x_0)$$

which forces the equality $f(2x_0) = 2f(x_0)$, as desired.

Ad (vii). Setting $y = x$ in (2), we obtain the equality $f(2x)f(x^2) = 2f(x)^3$ for all $x \in X$, whence, on account of (vi), we infer that

$$2f(x)f(x^2) = 2f(x)^3, \quad x \in X,$$

which implies (vii) whenever $x \neq 0$. However, for $x = 0$, we have $f(0^2) = f(0) = 0 = f(0)^2$, which completes the proof of property (vii).

Ad (viii). Fix arbitrarily elements x, y from X. Replacing y by $2y$ in (2) and applying (vi), we deduce the following equalities

$$2f(x+2y)f(xy) = f(x+2y)f(2xy) = f(x)^2 f(2y) = f(x)f(2y)^2 = 2f(x)^2 f(y) + 4f(x)f(y)^2,$$

whence

$$f(x+2y)f(xy) = f(x)^2 f(y) + 2f(x)f(y)^2 = (f(x)^2 f(y) + f(x)f(y)^2) + f(x)f(y)^2 = f(x+y)f(xy) + f(x)f(y)^2.$$

Consequently,

(a) $(f(x+2y) - f(x+y))f(xy) = f(x)f(y)^2,$

whence, by interchanging the roles of x and y, we obtain

(b) $(f(2x+y) - f(x+y))f(yx) = f(x)^2 f(y).$

Observe that $f(xy) = f(yx)$; actually, since the right-hand side of (2) is symmetric with respect to x and y, we obtain easily the relation $f(x+y)[f(xy) - f(yx)] = 0$, whence the desired equality results immediately provided that $f(x+y) \neq 0$, i.e., whenever $y \neq -x$. If $y = -x$, we have also

$$f(x(-x)) = f(-x^2) = f((-x)x).$$

Thus, adding the equalities (a) and (b) side by side, we arrive at

$$[f(x+2y) + f(2x+y) - 2f(x+y)]f(xy) = f(x)^2 f(y) + f(x)f(y)^2 = f(x+y)f(xy)\,,$$

i.e.,

$$[f(x+2y) + f(2x+y) - 3f(x+y)]f(xy) = 0\,.$$

Therefore, the following implication holds true:

$$f(xy) \neq 0 \Longrightarrow f(x+2y) + f(2x+y) = 3f(x+y)\,.$$

In the case where $f(xy) = 0$, equality (a) implies that $f(x)f(y)^2 = 0$, forcing x or y to vanish. If $x = 0$, then, by means of (vi), $f(x+2y) + f(2x+y) = f(2y) + f(y) = 3f(y) = 3f(x+y)$; if $y = 0$, we proceed analogously. This finishes the proof of property (viii).

Ad (ix). Put $y = x$ in (viii). Then, in view of (vi), we obtain

$$2f(3x) = 3f(2x) = 6f(x) \quad \text{whence} \quad f(3x) = 3f(x).$$

Thus, the proof of our Proposition 2 has been completed. □

4. The Main Result

Now, we are in a position to prove the following:

Theorem 1. *Let $f : X \to Y$ be a solution to Equation* (2) *such that $f(0) = 0$ if and only if $x = 0$. Then, f yields a ring homomorphism between X and Y.*

Proof. Fix arbitrarily elements s and t from X. Let $x := 2t - s$ and $y := 2s - t$. Then, $x + 2y = 3s$ and $2x + y = 3t$ and, in view of the properties (ix) and (viii) from Proposition 2, we have

$$3f(s) + 3f(t) = f(3s) + f(3t) = f(2x+y) + f(x+2y) = 3f(x+y) = 3f(s+t)$$

which shows that f is additive.

On the other hand, by means of the additivity of f just proven and the property (vii), we obtain

$$(f(s)+f(t))^2 = f(s+t)^2 = f((s+t)^2) = f(s^2+st+ts+t^2) = f(s^2)+f(st)+f(ts)+f(t^2),$$

whence

$$f(s^2) + 2f(s)f(t) + f(t^2) = f(s)^2 + 2f(s)f(t) + f(t)^2 = (f(s)+f(t))^2 = f(s^2)+f(st)+f(ts)+f(t^2).$$

Since $f(st) = f(ts)$ (see the proof of property (vii) from Proposition 2), the latter equality proves the multiplicativity of f and finishes the proof.
□

5. Three Corollaries

Our main result implies, almost immediately, what follows.

Corollary 1. *The only solution $f : \mathbb{R} \longrightarrow \mathbb{R}$ of Equation* (2) *such that* $\ker f = \{0\}$ *is the identity map. If* $\operatorname{card} \ker f \geq 2$, *then* $\ker f$ *is large.*

In fact, with $\ker f = \{0\}$, our Theorem states that f is a homomorphism. The multiplicativity of f implies then that $f(x^2) = f(x)^2 \geq 0$ and the additivity of $-f$, which is upper-bounded on a nonempty open set, forces $-f$ and hence also f to be linear: $f(x) = \alpha x, x \in \mathbb{R}$, with some nonzero coefficient $\alpha \in \mathbb{R}$. Applying the multiplicativity of f once again, we infer that $\alpha = 1$.

The latter assertion results from Proposition 1.

Corollary 2. *The only solution $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ of Equation* (2) *such that* $\ker f = \{0\}$ *is the identity map. If $e \in$* $\ker f$, *then* $\ker f$ *is large.*

Indeed, with $\ker f = \{0\}$, our theorem states that f is a homomorphism. Since $f(1) = 1$ (cf. property (ii) from Proposition 2), the additivity of f alone gives the claim $f(n) = n$, $n \in \mathbb{Z}$.

Moreover, in the present case, the latter assertion results from Proposition 1.

Corollary 3. *If* $f : X \longrightarrow Y$ *yields a solution of Equation* (2) *such that* $e \in \ker f$ *and* 2 *is invertible in* Y, *then* $\ker f$ *is large. In particular, a function*

$$f(x) = \begin{cases} 0 & for\, x \in X \setminus Z \\ \frac{1}{2} & for\, x \in Z \end{cases}$$

where Z *stands for a subring of* X *with* $\operatorname{card} X/Z = 2$, *solves Equation* (2).

Actually, the largeness of $\ker f$ results from Proposition 1. Note that setting $Z' := X \setminus Z$, we have $Z + Z \subset Z$, $Z \cdot Z \subset Z$ and $Z' + Z = Z + Z' \subset Z'$. Now, a straightforward verification proves the latter assertion.

6. An Exotic Example

Let X be the power set 2^T of a set $T = \{a, b\}, a \neq b$. With binary operations $A + B := A \cup B$ and $A \cdot B := A \cap B$, $(A, B) \in X \times X$, the set X becomes a commutative unitary ring with $0 = \emptyset$ and the unit $e := X$. If $f : X \longrightarrow X$ is a solution to Equation (2) such that $\ker f = \{\emptyset\}$; then, according to the Theorem, f yields an isomorphism of X onto itself, i.e., $f(A) = A$ for every set $A \subset X$, or $f(0) = 0, f(e) = e, f(\{a\}) = \{b\}$ and $f(\{b\}) = \{a\}$.

Clearly, we can extrapolate this example to larger finite sets T, obtaining the identity map as a suitable solution of (2) up to a permutation of elements of T.

7. Final Remarks

Looking back at the statement of Proposition 1, we may summarize it in the following way: the occurrence of nonhomomorphic solutions of Equation (2) may emerge if we deal with functions with small ranges and/or large sets of zeros only. Our Proposition 1 does not provide any precise description of the analytic shape of such solutions. Although, personally, I see the need to obtain such a description as a problem of minor significance, I plan to address this in another article, because I believe that such a task requires some persistence.

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