PRACE NAUKOWE UNIWERSYTETU ŚLĄSKIEGO W KATOWICACH NR 30
PRACE MATEMATYCZNE III, 1973

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ON GENERALIZED SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF ORDER n

§ 1. INTRODUCTION. This paper deals with the following differential equation:

$$x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \dots + p_n(t)x(t) + p_{n+1}(t) = 0, \tag{*}$$

in which $p_i(t)$ for $i = 1, \dots, n+1$ are know measures in the spaces R^1, unknown is the distribution $x(t)$. The derivative is understood in the distribution sense.

In this paper we shall prove the theorem on the existence of a unique solution of Cauchy problem for the equation (*) in the class all distributions, which $(n-1)$ derivatives in the distribution sense can be identified with functions of finite variations in the space R^1. The following theorem generalized some result of A. Lasota and F.H. Szafraniec (see [8]). Our theorems can be aplied to some equations for which the theorems of J. Kurzweil (see [7]) cannot be used.

Principal result of this paper is based on the sequential theory of the distribution (see [9]).

§ 2. NOTATIONS.

Definition 1. A sequence of smooth and non-negative functions $\delta_n(t)$ satisfying:

1° $\displaystyle\int_{-\infty}^{\infty} \delta_n(t)\,dt = 1$

2° There is a sequence of positive numbers a_n convergent to zero such that

$$\delta_n(t) = 0 \text{ for } |t| \geqslant a_n$$

3° There are numbers $M_o, M_1, \dots$ such that

$$a_n^k \int_{-\infty}^{\infty} |\delta_n^{(k)}(t)|\, dt < M_k$$

holds for $n = 1,2, \ldots$ and every order k.

4° $$\delta_n(t) = \delta_n(-t)$$

will be called the delta sequence.

The condition 4° was suggested by P. Antosik (see [4]).

Definition 2. By a regular sequence for a distribution f we understand any sequence the n — therm of which is (see [10]):

$$f_n = f^* \delta_n.$$

Definition 3. If for every regular sequence $f_n(x)$ for distribution $f(x)$ the sequence $f_n(x_o)$ is convergent, then the limit $\lim\limits_{n\to\infty} f_n(x_o)$ will be called the mean value of distribution $f(x)$ in the point x_o and denoted by $f(x_o)$.

This definition of the value of distribution in the point may be found in [4].

Definition 4. We say that a distribution f is a measure, if there exists for f a fundamental sequence f_n such that, to each finite interval I, the sequence of numbers $\int_I |f_n|$ is bounded (see [2]).

Definition 5. By a solution of equation (*) we understand any distribution $x(t)$, which satisfies this equation in the space R^1.

Definition 6. We shall denote by V^{n-1} a class of all distributions, which $(n-1)$ derivatives in the distribution sense can be identified with functions of finite variations in the space R^1.

§ 3. LEMMAS. We shall prove two lemmas.

LEMMA 1. *If distribution $P(t)$ is a function of finite variation in R^1, then there exists a value of $P(t)$ in every point $t_o \in R^1$ in the sense of definition 3 and*

$$P(t_o) = \frac{1}{2}[P(t_o^+) + P(t_o^-)],$$

where $P(t_o^+)$, $P(t_o^-)$ denote respectively right and left limits of a fnuction $P(t)$ in t_o (see [4]).

Proof. From the definition 3 and condition 4° for delta sequence we have:

$$P(t_o) = \lim_{n\to\infty} \Big[\int_{-an}^{0} P(t-\tau)\delta_n(\tau)d\tau + \int_{0}^{an} P(t-\tau)\, \delta_n(\tau)\, d\tau\Big]_{t=to} = \lim_{n\to\infty} \Big[\int_{0}^{an} (P(t+s) +$$

$$+ P(t-s))\, \delta_n(s)\, ds\Big]_{t=to} = \lim_{n\to\infty} \Big[(P(t^+) + P(t^-)) \int_{0}^{an} \delta_n(s)\, ds\Big]_{t=to},$$

which ends the proof of the lemma.

LEMMA 2. *If $p_n(t)$ is an arbitrary regular sequence for measure $p(t)$ and $x_n(t)$ an arbitrary sequence of smoth functions almost uniformly convergent to a function $x(t)$, then*

$$\lim_{n\to\infty}(d)\ p_n(t)x_n(t) = p(t)x(t),$$

where the convergence is understood in the distribution sense (what is denoted by lim(d)).

Proof. Let $\bar{x}_n(t)$ be an arbitrary regular sequence for the continuous function $x(t)$. Then for an arbitrary number $\varepsilon > 0$ there exists such a number n_o that for every number $n > n_o$ we have:

$$\left|\int_a^t (x_n(s)-\bar{x}_n(s))\, p_n(s)\, ds\right| \leqslant \varepsilon \left|\int_a^t |p_n(s)|\, |ds|\right|.$$

This last inequality finished the proof.

§ 4. PRINCIPAL RESULT.

THEOREM. *Let $p_i(t)$ for $i = 1, \ldots, n+1$ be measures and $p_1(t)$ a locally integrable function in the space R^1. Then the problem*

$$\begin{cases} x^{(n)}(t) + p_1(t)\, x^{(n-1)}(t) + \ldots + p_n(t)\, x(t) + p_{n+1}(t) = 0 \\ x^{(l)}(a) = \varkappa_l,\ l = 0, \ldots, n-1 \end{cases} \tag{**}$$

has exactly one solution in the class V^{n-1} ($x^{(l)}(a)$ is the mean value of distribution $x^{(l)}(t)$ in a).

At first we shall prove two lemmas.

LEMMA 3. *Let $p_{ik}(t)$ be arbitrary regular sequence for the measures $p_i(t)$ for $i = 1, \ldots, n+1$ and $\varkappa_{lk}$ for $l = 0, \ldots, n-1$ be arbitrary convergent sequences respectively to $\varkappa_l$ as $k\to\infty$. Then sequences $x_k^{(l)}(t)$ defined by*

$$x_k^{(l)}(t) = -\left[\int_a^t (t-s)^{n-1} \left(\sum_{i=1}^{n} p_{ik}(s)\, x_k^{(n-i)}(s) + p_{n+1k}(s)\right) ds\right]^{(l)} + \\ + \left[\sum_{\mu=0}^{n-1} \varkappa_{\mu k} \frac{(t-a)^{\mu}}{\mu!}\right]^{(l)} \tag{1}$$

are locally equibounded in R^1.

Proof. Replacing equation

$$x^{(n)}(t) = -\left[\sum_{i=1}^{n} p_{ik}(t)\, x^{(n-i)}(t) + p_{n+1k}(t)\right] \tag{2}$$

by a system of equations we have the following estimation (see [6] p. 72):

$$\|Y_k(t)\| \leqslant \left\{\|Y_k(a)\| + \left|\int_a^t \|B_k(s)\| ds\right|\right\} exp\left|\int_a^t \|A_k(s)\| ds\right|, \tag{3}$$

where

$$Y_k(t) = \begin{bmatrix} x_k(t) \\ x_k'(t) \\ \cdot \\ \cdot \\ \cdot \\ x_k^{(n-1)}(t) \end{bmatrix}, \quad Y_k(a) = \begin{bmatrix} \varkappa_{0k} \\ \varkappa_{1k} \\ \cdot \\ \cdot \\ \cdot \\ \varkappa_{n-1k} \end{bmatrix}, \quad B_k(t) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ -p_{n+1k}(t) \end{bmatrix},$$

$$A_k(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 0 & . & . & . & . \\ . & . & . & . & . & . & . & 1 \\ -p_{nk}(t) & . & . & . & . & . & . & -p_{1k}(t) \end{bmatrix},$$

$$x_k^{(n)}(t) = -\left[\sum_{i=1}^{n} p_{ik}(t)\, x_k^{(n-i)}(t) + p_{n+1k}(t)\right],$$

the symbol $\|A\|$ denotes the sum of the absolute values of all elements of the matrix A. From inequality 3 it is easy to prove our lemma.

LEMMA 4. If the assumptions of lemma 3 are satisfied, then there exist subsequences $\{x_{rk}^{(l)}(t)\}$ of $\{x_k^{(l)}(t)\}$ convergent to a functions $x^{(l)}(t)$ of finite variations in R^1.

P r o o f. From lemma 3 and the definition 4 for each finite interval I we have:

$$\sum_{m=1}^{p} | x_k^{(l)}(t_m) - x_k^{(l)}(t_{m-1}) | < \infty,$$

where p is an arbitrary natural number and $t_\gamma \in I$, $\gamma = 0, \ldots, p$. From Helly's theorem (see [12] p. 372) it follows that some subsequences $x_{ik}^{(l)}(t)$ of $x_k^{(l)}(t)$ are convergent to $x^{(l)}(t)$ as $t \in I$ and $l = 0, \ldots, n-1$. Let a_λ be an arbitrary decreasing sequence, b_λ strongly increasing sequence such that

$$I = [a, b] = [a_1, b_1], \lim_{\lambda \to \infty} a_\lambda = -\infty, \lim_{\lambda \to \infty} b_\lambda = \infty,$$

and $x_{pk}^{(l)}(t)$ subsequences of $x_{p-1k}(t)$ for $p = 2, 3, \ldots$ and $l = 0, \ldots n-1$ and that their limits are functions of finite variations in $[a_p, b_p]$. Let us consider the matarix

$$\begin{bmatrix} x_{11}^{(l)}(t) & x_{12}^{(l)}(t) & \ldots & x_{1k}^{(l)}(t) & \ldots \\ \cdot & \cdot & \ldots & \cdot & \ldots \\ x_{r1}^{(l)}(t) & x_{r2}^{(l)}(t) & \ldots & x_{rk}^{(l)}(t) & \ldots \\ \cdot & \cdot & \ldots & \cdot & \ldots \end{bmatrix}.$$

The diagonal sequences $x_{kk}^{(l)}(t)$ are convergent to some functions $x^{(l)}(t)$ of finite variations in R^1, which completes the proof of lemma 4.

Without loss of generality we can assume that

$$x_{kk}^{(l)}(t) = x_k^{(l)}(t).$$

Then considering that $p_1(t)$ is locally integrable function by lemma 2 and [3] (p. 642) we obtain

$$\lim_{k\to\infty} (d)\,[x_k^{(n)}(t) + p_{1k}(t)\,x_k^{(n-1)}(t) + \ldots + p_{nk}(t)\,x_k(t) + p_{n+1k}(t)] =$$
$$= [x^{(n)}(t) + p_1(t)\,x^{(n-1)}(t) + \ldots + p_n(t)\,x(t) + p_{n+1}(t)].$$

Obviously $x^{(l)}(a) = \varkappa_l$ for $l = 0, \ldots, n-2$. We shall prove that $x^{(n-1)}(a) = \varkappa_{n-1}$. In fact, let $f_{1k}, \ldots, f_{nk}$ be regular sequences formeasures $p_1(t) \cdot x^{(n-1)}(t), \ldots, p_n(t) \cdot x(t)$ respectively. The sequence

$$Y_k(t) \overset{\mathrm{df}}{=} -\Big[\int_a^t f_{1k}(s) + \ldots + f_{nk}(s) + p_{n+1k}(s)\Big]\,ds + \varkappa_{n-1}$$

is convergent to a function $Y(t)$ of finite variation. By conditions (* *) we obtain $Y(t) = x^{n-1}(t)$. Then from definition 3 and lemma 1 we have:

$$x^{(n-1)}(a) = -\lim_{\eta\to\infty}\,[\lim_{k\to\infty}\int_{-\infty}^{\infty}\Big(\int^{t-\tau}(f_{1k}(s) + \ldots + p_{n+1k}(s))\,ds\Big)\,\delta_\eta(\tau)d\tau]_{t=a} + \varkappa_{n-1} =$$
$$= -\lim_{k\to\infty}\int_{-\infty}^{-\infty}[F_1(t-\tau) - F_1(a) + \ldots + P_{n+1}(t-\tau) - P_{n+1}(a)]\delta\eta(\tau)\int|_{t=a} +$$
$$+ \varkappa_{n-1} = \varkappa_{n-1},$$

where it was assumed, that

$$F'_i(t) = f_i(t) = p_i(t)\,x^{(n-1)}(t),\quad P'_{n+1}(t) = p_{n+1}(t),\quad i = 1, \ldots, n.$$

Now we shall prove the uniqueness of the solution of problem (* *) in the class V^{n-1}. *Suppose that* $x_1(t) \in V^{n-1}$, $x(t) \in V^{n-1}$ *are solutions of* problem (* *) and $x_1(t) \neq x_1(t)$. Let us denote by $x_{1k}(t)$, $x_k(t)$ regular sequences for $x_1(t)$ *and* $x(t)$ respectively. We consider $\{Y_{1k}^{(l)}(t)\}$, $\{Y_k^{(l)}(t)\}$ for $l = 0, \ldots, n-1$.

$$Y_{1k}^{(l)}(t) \overset{\mathrm{df}}{=} -\Big[\int_a^t (t-s)^{n-1}\Big(\sum_{i=1}^{n} p_{ik}(s)\overset{(n-i)}{x_{1k}}(s) + p_{n+1k}(s)\Big)\,ds\Big]^{(l)} + \left(\sum_{\mu=0}^{n-1}\varkappa\mu\,\frac{(t-a)^\mu}{\mu!}\right)^{(l)}. \tag{4}$$

$$Y_k^{(l)}(t) \overset{\mathrm{df}}{=} -\Big[\int_a^t (t-s)^{n-1}\Big(\sum_{i=1}^{n} p_{ik}(s)\overset{(n-i)}{x_k}(s) + p_{n+1k}(s)\Big)\,ds\Big]^{(l)} + \left(\sum_{\mu=0}^{n-1}\varkappa\mu\,\frac{(t-a)^\mu}{\mu!}\right)^{(l)}. \tag{5}$$

The sequences $Y_{1k}^{(l)}$, $Y_k^{(l)}$ are locally equibound in the space R^1 and there exist subsequences of these sequences convergent to some function $Y_1^{(l)}(t)$ and $Y^{(l)}(t)$ of finite variations. Without loss of generality we can assume that the sequences $Y_{1k}^{(l)}(t)$ and $Y_k^{(l)}(t)$ are convergent to functions $Y_1^{(l)}(t)$, $Y^{(l)}(t)$ (for $l < n-1$ it is an almost uniform convergence). By lemma 2, (* *), [3] (p. 642), [9] (p. 21) we have:

$$Y_1^{(l)}(t) = x_1^{(l)}(t), \; Y^{(l)}(t) = x^{(l)}(t).$$

Let's put

$$\varepsilon_{lk}(t) = \| x_{1k}^{(l)}(t) - x_k^{(l)}(t) | - | Z_k^{(l)}(t) \|, \tag{6}$$

where

$$|Y_{1k}^{(l)}(t) - Y_k^{(l)}(t)| = |Z_k^{(l)}(t)|, \; l = 0, \ldots, n-1.$$

A sequence $\varepsilon_{lk}(t)$ for $l < n-1$ and $k \to \infty$ is almost uniformly convergent to zero in R^1. The sequence $\varepsilon_{n-1k}(t)$ for $k \to \infty$ tends to zero in the distribution sense (it is locally equibounded and almst everywhere convergent to zero). From (4), (5) we have:

$$|Z_k^{(n)}(t)| \leqslant \sum_{i=1}^{n} |p_{ik}(t)| |Z_k^{(n-i)}(t)| + \bar{B}_k(t), \tag{7}$$

where

$$\bar{B}_k(t) \overset{df}{=} \varepsilon_{ok}(t) |p_{nk}(t)| + \ldots + \varepsilon_{n-1k}(t) | p_{1k}(t)|.$$

Replacing inequality (7) by a system of inequalities we obtain by [6] (p. 39—40, 73):

$$\|U_k(t)\| \leqslant \{\|U_k(a)\| + | \int_a^t B_k(s) ds | \} \exp | \int_a^t \|A_k(s)\| \, ds |, \tag{8}$$

where

$$U_k(t) = \begin{bmatrix} Z_k(t) \\ Z_k'(t) \\ \cdot \\ \cdot \\ \cdot \\ Z_k^{(n-1)}(t) \end{bmatrix}, U_k(a) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \; \bar{A}_k(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ |p_{nk}| & \cdot & \cdot & \cdot & \ldots & |p_{1k}| \end{bmatrix}$$

From inequality (8) and [3] (p. 642) it follows that $x_1(t) = x(t)$, what completes the proof of the theorem.

R e m a r k 1. The theorem above generalizes some results of *A.* Lasota and F. H. Szafraniec (see [8]). Our theorem can be applied to some equations for which the theorems of J. Kurzweil [7] cannot be used. One of such equation is

$$x^{(n)}(t) + x^{(n-1)}(t) + \delta(t) x^{(n-2)}(t) + \ldots + \delta(t) x(t) + \delta(t) = 0,$$

(where $\delta(t)$ denotes Dirac's delta).

Remark 2. Let's consider the equation

$$x'(t) = -\frac{1}{2t} x(t)$$

with the initial condition $x(1) = 0$. It is known that the last problem has two solutions $x_1(t) = 0$ and $x_2(t) = \delta(t)$ (see [5]). This example shows that without additional conditions the problem of Cauchy can have more than one solution.

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O ROZWIĄZANIACH UOGÓLNIONYCH RÓWNAŃ LINIOWYCH n-TEGO RZĘDU

Streszczenie

W pracy tej udowodniono twierdzenie o istnieniu i jednoznaczności rozwiązania problemu

$$\begin{cases} x^{(n)}(t) + p_1(t)\, x^{(n-1)}(t) + \ldots + p_n(t)\, x(t) + p_{n+1}(t) = 0 \\ x^{(l)}(a) = \varkappa_l,\ l = 0, \ldots, n-1 \end{cases}$$

w klasie dystrybucji, których $(n-1)$ pochodne w sensie dystrybucyjnym można utożsamić z funkcjami o wahaniu skończonym w przestrzeni R^1, przy czym $p_i(t)$ dla $i = 1, \ldots, n+1$ oznaczają dane miary (por. [2]), zaś $x^{(l)}(a)$ oznacza średnią wartość dystrybucji $x^{(l)}(t)$ w punkcie a (por. [4]). Powyższe twierdzenie uogólnia pewien wynik A. Lasoty i F. H. Szafrańca (por. [8]). Ponadto twierdzenie to może być zastosowane do pewnych typów równań, dla których twierdzenia J. Kurzweila (por. [7]) nie mogą być stosowane

Zasadniczy wynik tej pracy jest oparty na ciągowej teorii dystrybucji (por. [9]).

Oddano do Redakcji 25.11.71.