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ADAM BIELECKI, TADEUSZ DŁOTKO

ON THE CURL OF SINGULAR COMPLETELY CONTINOUS VECTOR FIELDS IN BANACH SPACES

Let us consider the Banach space X with the sphere S $\{x : x \in X, \|x\| = R\}$. By $h(x)$ we denote the application of this sphere into X and assume that h is completely continous. The application $\Phi(x) = x - h(x)$ is called a completely continous vector field on $\|x\| = R$.

The condition $\Phi(x) \neq 0$ is essential in the definition of the curl $\gamma(\Phi, R)$ of the vector field (see [1] p. 112).

A vector field $\Phi(x)$ on $\|x\| = R$ is called non singular on S if $\Phi(x) \neq 0$ for $\|x\| = R$.

But in special cases it is admissible to introduce $\gamma(\Phi, R)$ for singular completely continous vector fields. Therefore let us change the definition of a curl $\gamma(\Phi, R)$ in a suitable manner.

It is easy to prove that if $\Phi(x) = x - h(x)$, $\psi(x) = x - H(x)$ are two completely continous vector fields on the sphere $\|x\| = R$, $\Phi(x)$ is non singular on S and

$$\|h(x) - H(x)\| < \|x - h(x)\| \text{ for } \|x\| = R, \tag{1}$$

then $\psi(x)$ is also non singular on S and $\gamma(\psi, R) = \gamma(\Phi, R)$ (see [1] p. 170). In fact, if

$$H_\lambda(x) = \lambda H(x) + (1-\lambda) h(x), \quad 0 \leqslant \lambda \leqslant 1, \quad \|x\| = R,$$

then

$$\|x - H_\lambda(x)\| = \|x - \lambda H(x) - (1-\lambda) h(x)\| \geqslant$$
$$\geqslant \|x - h(x)\| - |\lambda| \, \|H(x) - h(x)\| >$$
$$> \|x - h(x)\| (1-\lambda) \geqslant 0,$$

so

$$x - H_\lambda(x) \neq 0 \text{ for } \|x\| = R, \ \lambda \in [0,1].$$

In particular for $\lambda = 1$ and $\|x\| = R$; $x - H(x) \neq 0$. We proved that for

$\lambda \in [0,1]$ the integer number function $\Gamma(\lambda) = \gamma(H_\lambda(x), R)$ is defined and continous. Just for that reason $\Gamma(\lambda) = \text{const}$. We have $\gamma(h(x), R) = \gamma(H_\lambda(x), R) = \gamma(H(x), R)$, $\lambda \in [0,1]$.

If we replace (1) by a weakly condition

$$\| H(x) - h(x) \| \leqslant \| x - h(x) \|, \tag{2}$$

then we have $\| x - H(x) \| > 0$ for $\lambda \in [0,1)$, because $\| x - H_\lambda(x) \| \geqslant$

$$\geqslant (1 - \lambda) \| x - h(x) \| > 0.$$

From this considerations we have a natural generalisation

$$\gamma(H, R) = \gamma(H_1, R) = \gamma(h, R)$$

DEFINITION. If $\psi(x) = x - H(x)$ is a completely continous vector field on $\| x \| = R$ and if there exist a completely continous vector field $\Phi(x) = x - h(x)$ non singular and satisfied (2), then

$$\gamma(H, R) \overset{\text{df}}{=} \gamma(h, R).$$

It is interesting, that by such generalization of the curl $\gamma(H, R)$ the following theorem is true.

THEOREM. If the completely continous vector field

$$\psi(x) = x - H(x)$$

defined for $\| x \| \leqslant R$ has the generalized curl $\gamma(H, R)$ and

$$\gamma(H, R) \neq 0 \quad \text{for} \quad \| x \| = R,$$

then the equation

$$x = H(x)$$

has at the least one solution x_0 such, that $\| x_0 \| \leqslant R$.

Demonstration. In accord with our assumptions it exist a absolutely continous function $h(x)$ such, that

$$\| H(x) - h(x) \| \leqslant \| x - h(x) \| \text{ and } \| x - h(x) \| > 0 \text{ for } \| x \| = R.$$

Let us define two prolongations h^* and H^* of h and H on $\| x \| = R$ as follows

$$h^*(x) = \begin{cases} h(x) & \text{for} \quad \| x \| \leqslant R \\ \dfrac{\| x \|}{R} h\left(\dfrac{Rx}{\| x \|}\right) & \text{for } \| x \| \geqslant R \end{cases}$$

and

$$H^*(x) = \begin{cases} H(x) & \text{for} \quad \| x \| \leqslant R \\ \left(\dfrac{\| x \|}{R} - 1\right) h\left(\dfrac{Rx}{\| x \|}\right) + H\left(\dfrac{Rx}{\| x \|}\right) & \text{for } \| x \| \geqslant R. \end{cases}$$

The functions h^* and H^* are completely continuous.
Let $\| x \| = R + \varepsilon,\ \varepsilon > 0$, then

$$\| H^*(x) - h^*(x) \| \leqslant \| R \frac{x}{\| x \|} - h\left(\frac{x}{\| x \|} R\right) \| =$$

$$= \frac{R}{\| x \|} \| x - \frac{\| x \|}{R} h\left(\frac{Rx}{\| x \|}\right) \| < \| x - h^*(x) \|.$$

If $\varepsilon_n = 1/n$, then from considerations mentioned above it exist a sequence x_n, $\| x_n \| \leqslant R + 1/n$ and

$$x_n = H^*(x_n).$$

Let us consider the sets $\{x_n\}$ and $\{H^*(x_n)\}$ for $n = 1,2,\ldots$
The last set is compact, so we can take a subsejuence $H^*(x_n)$ which tend to x_o as $k \to \infty$.
So we have

$$H^*(x_{n_k}) = x_{n_k}$$

and for $n \to \infty$

$$H^*(x_o) = x_o.$$

But $\| x_o \| \leqslant R$, and $H^*(x_o) = H(x_o) = x_o$ which finished the demonstration.

REMARK. Let us assume that $\Phi(x) = x - h(x)$ and $\psi(x) = x - H(x)$ are two completely continous vector fields on $\| x \| = R$, $\Phi(x)$ is non singular and $\gamma(h, R) \neq 0$. If exist such an element $\bar{x}$, $\| \bar{x} \| = R$ that

$$H(\bar{x}) = 2\, h(\bar{x}) - \bar{x},$$

then from condition (1) does not result that $\gamma(H, R)$ exist and

$$\gamma(H, R) = \gamma(h, R).$$

Instead if we assume the generalized definition od $\gamma(h, R)$ and condition (2) is true, then from the theorem above

$$\gamma(H, R) = \gamma(h, R).$$

REFERENCES.

[1]. M. A. Krasnosielskij: *Topological methods in the theory of nonlinear integral equations.* Moskwa 1956 (russ.).

[2]. M. A. Krasnosielskij, P.P. Zabreiko, J.B. Rutickij, W.J Otjecenko: *Aproximate solution of operator equations.* Moskwa 1969 (russ.).

A. BIELECKI ET T. DŁOTKO

SUR LA ROTATION DU CHAMP VECTORIEL SINGULIER COMPLÈTEMENT CONTINU DANS L'ESPACE DE BANACH

Résumé

Dans cette note on considerè un champ vectoriel $\varphi(x) = x - F(x)$ pour $\|x\| = R > 0$ de l'espace de Banach X, et l'opération $F: X \to X$ est complètement continue.

Il peut arriver, qu'il existe $x \in X$, $\|x\| = R$ et $\varphi(x) = 0$.

On généralise la définition de la rotation du champ vectoriel complètement continu et non singulier en un champ singulier, en conservant le théorème fondamental: Si la rotation généralisée est différente de zéro sur la sphère $\|x\| = R$, alors l'équation $x = H(x)$ possède au moins une solution, telle que $\|x\| \leqslant R$.

Cela consiste un supplément à la théorie des champs vectoriels complètement continus.

ADAM BIELECKI I TADEUSZ DŁOTKO

O OBROCIE OSOBLIWEGO PEŁNOCIĄGŁEGO POLA WEKTOROWEGO W PRZESTRZENI BANACHA

Streszczenie

W pracy rozpatruje się pole wektorowe $\varphi(x) = x - F(x)$ na sferze $\|x\| = R > 0$ przestrzeni Bancha X, przy czym operacja $F: X \to X$ jest pełnociągła.

Dopuszcza się istnienie miejsc zerowych pola φ na sferze $\|x\| = R$.

Uogólnia się definicję obrotu pełnociągłego nieosobliwego pola wektorowego na pole osobliwe, przy tym zachowuje się podstawowe twierdzenie: Jeżeli uogólniony obrót jest różny od zera na sferze $\|x\| = R$, to równanie $x = H(x)$ ma przynajmniej jedno rozwiązanie zawarte w kuli domkniętej $\|x\| \leqslant R$.

Stanowi to uzupełnienie teorii pełnociągłych pól wektorowych.