Title: On the dependence on a parameter of solutions of a linear functional equation

Author: Stefan Czerwik

PRACE NAUKOWE UNIWERSYTETU ŚLĄSKIEGO W KATOWICACH NR 30
PRACE MATEMATYCZNE III, 1973

STEFAN CZERWIK

ON THE DEPENDENCE ON A PARAMETER OF SOLUTIONS OF A LINEAR FUNCTIONAL EQUATION

1. INTRODUCTION. In the present paper we are concerned with the linear functional equation

$$\varphi[f(x,t)] = g(x,t)\,\varphi(x) + F(x,t), \tag{1}$$

where $\varphi(x)$ is an unknown function and $f(x,t)$, $g(x,t)$, $F(x,t)$ are known real functions of real variables. The variable t is regarded as a parameter.

We shall prove that under some assumptions concerning the given functions $f(x,t)$, $g(x,t)$, $F(x,t)$, the solution $\varphi(x,t)$ of equation (1) which is continuous with respect to x is also continuous with respect to the couple of variables x, t.

For the natural parameter the continuous dependence of solutions of equation (1) on given functions has been investigated in [4], [2], and for the more general equation

$$\varphi(x) = H_n(x, \varphi[f_n(x)])$$

in [5].

2. Let us introduce the notations:

$$f^0(x,t) = x, \quad f^{n+1}(x,t) = f[f^n(x,t), t], \quad n = 0, 1, \ldots,$$

$$G_n(x,t) = \prod_{\nu=0}^{n-1} g[f^\nu(x,t), t], \tag{2}$$

$$\overline{F}(x,t) = F(x,t) + c(t)\,[g(x,t) - 1]. \tag{3}$$

The functions f, g, F will be subjected to the following conditions:

(*i*) The function $f(x,t)$ is continuous in $\Delta = \langle(a,b) \times T$, where T is an interval (finite or not), $a < f(x,t) < x$ in $(a,b) \times T$, $f(a,t) = a$ for $t \in T$ and, for every fixed $t \in T$, f is strictly increasing in (a,b).

(*ii*) The function $g(x,t)$ is continuous in Δ, $g(x,t) \neq 0$ in Δ.

(*iii*) The function $F(x,t)$ is continuous in Δ.

(*iv*). There exists a function $c(t)$ such that for every closed interval $<\alpha, \beta> \subset T$ there exist an interval $< a, a+d> \subset < a, b>$, $d > 0$, a function $B(x, t)$ continuous with respect to the variable t in $<\alpha, \beta>$ and bounded in $< a, a+d> \times <\alpha, \beta>$, and a constant $0 < \Theta < 1$ such that the inequalities

$$(4) \qquad |F(x, t)| \leqslant B(x, t),$$

$$(4') \qquad B[f(x, t), t] \leqslant \Theta B(x, t)$$

hold in $< a, a+d> \times <\alpha, \beta>$.

(*v*) There exists a constant $K > 0$ such that

$$(5) \qquad \frac{1}{G_n(x, t)} \leqslant K \text{ for } (x, t) \in \Delta \text{ and } n = 1,2, \ldots .$$

For a fixed $t \in T$ there are the following three possibilities:
(A) The limit

$$(6) \qquad G(x, t) = \lim_{n \to \infty} G_n(x, t)$$

exists, $G(x, t)$ is continous and $G(x, t) \neq 0$ in $< a, b)$.
(*B*) There existts an interval $J \subset (a, b)$ such that $\lim_{n \to \infty} G_n(x, t) = 0$ uniformly in J.
(C) Neither (*A*) nor (*B*) occurs.
Now we shall prove

THEOREM 1 *Suppose that hypotheses (i)—(v) are fulfilled. If, moreover, for every $t \in T$ case* (C) *occurs, then the solution $\varphi(x, t)$ of equation* (1) *which is continuous with respect to x, is also continuous with respect to the couple of variables x, t in Δ. It is given by the formula*:

$$(7) \qquad \varphi(x, t) = \varphi_o(x, t) + c(t),$$

where

$$(8) \qquad \varphi_o(x, t) = - \sum_{v=0}^{\infty} \frac{F[f^v(x, t), t]}{G_{v+1}(x, t)} .$$

P r o o f. On account of theorem 9 in [1] for every fixed $t \in T$ there exists exactly one function $\varphi(x, t)$ satisfying equation (1) and continuous in $< a, b)$. It is given by formulas (7) and (8).

Let $< a, p> \times <\alpha, \beta> = I \subset \Delta$ be arbitrarily fixed. In view of (*i*) we have $\lim f^n(x, t) = a$ uniformly in I, hence there is an integer N such that

$$(9) \qquad f^n(x, t) \leqslant a + d \text{ for } n \geqslant N \text{ and } (x, t) \in I.$$

According to (*iv*, (*v*), (9) we have have for $v \geqslant N$ and $(x, t) \in I$

$$\left|\frac{F[f^v(x,t),t]}{G_{v+1}(x,t)}\right| \leqslant K B [f^v (x, t), t] \leqslant K \Theta^{v-N} B [f^N (x, t)] \leqslant K \Theta^{v-N} \sup B (x, t).$$

This shows that series (8) uniformly converges in I and, since I has been arbitrary, the function $\varphi_o (x, t)$ is continuous in A.

Now we shall prove that the function $c (t)$ is continuous in T. Let $t_o \in T$ be fixed and let $\{t_n\}$ $(n = 1,2, \dots)$ be any sequence such that $t_n \in T$ and $t_n \to t_o$. Since case (C) occurs, there exists a point $x_o \in \langle a, a + d \rangle$ such that $g (x_o, t_o) \neq 1$, and by (*ii*) there exists a number $M > 0$ such that $g (x_o, t_o) \neq 1$ for $n > M$. In view of (3) and (4)

$$| F (x_o, t_n) + c (t_n) [g (x_o, t_n) - 1] | \leqslant B (x_o, t_n),$$

and since $\{F (x_o, t_n)\}$ and $\{g (x_o, t_n)\}$ converge (the latter to $g (x_o, t_o) \neq 1$), the condition $g (x_o, t_n) \neq 1$ for $n > M$ implies the boundedness of the sequence $\{c (t_n)\}$.

Let us suppose that $\{c (t_n)\}$ does not converge; then there exist increasing sequnces of integers $\{n_k\}$ and $\{n_v\}$ such that

$$c(t_{n_k}) \underset{k\to\infty}{\rightarrow} s \text{ and } c(t_{n_v}) \underset{v\to\infty}{\rightarrow} q, \quad s \neq q,$$

and consequently

$$| F (x, t_o) + s [g (x, t_o) - 1] | \leqslant B (x, t_o),$$
$$| F (x, t_o) + q [g (x, t_o) - 1] | \leqslant B (x, t_o).$$

Hence we conclude that equation (1) has for $t = t_o$ two different solutions which is imposible (cf. [1], theorem 9). Since t_o has been arbitrary, the function $c (t)$ is continuous in T. This completes the proof of the theorem.

We shall show by an example that the conditions (*iv*) in theorem 1 is essential.

Example. Take $\langle a, b) = \langle 0,1)$, $T = \langle 0,1 \rangle$ and consider the equation

$$\varphi \left(\frac{x}{x+1}\right) = (1 + x) \varphi (x) - x + x^{t+1}. \tag{10}$$

We have

$$G_n (x, t) = \prod_{v=0}^{n-1} \left(1 + \frac{x}{1 + vx}\right), \quad G_n (x, t) \geqslant 1,$$

If $t \in (0,1)$, the series

$$\sum_{n=0}^{\infty} \frac{F [f^n (x, t), t]}{G_{n+1} (x, t)}$$

converges if $c (t) \equiv 1$ (cf. [1]), whence

$$\text{for } t > 0 : \varphi(x, t) = 1 - \sum_{n=0}^{\infty} \frac{F[f^n(x, t), t]}{G_{n+1}(x,t)} \text{ and } \varphi(0, t) = 1,$$

for $t = 0 : \varphi(x, 0) = 0$ and $\varphi(0, 0) = 0$,

and consequently $\varphi(x, t)$ is not continuous at $t = 0$, whence, in view of theorem 1, condition (*iv*) cannot be fulfilled.

3. We put

$$H_n(x, t) \overset{\text{df}}{=} \sum_{i=0}^{n-2} \prod_{v=i+1}^{n-1} g[f^v(x, t), t] \cdot F[f^i(x, t)\, t]. \tag{12}$$

Let $x_0 \in (a, b)$ be arbitrarily fixed. Put

$$\delta_0 \overset{\text{df}}{=} \{(x, t) : x \in \langle f(x_0, t), x_0 \rangle,\ t \in T\}. \tag{13}$$

Suppose that:

(*vi*) For every interwal $\langle \alpha, \beta \rangle \subset T$ there exists an $\bar{x}_0 \in a, b)$ such that

$$\lim_{n\to\infty} G_n(x, t) = \lim_{n\to\infty} H_n(x, t) = 0$$

uniformly in $\delta = \{(x, t) : x \in \langle f(\bar{x}_0, t), \bar{x}_0 \rangle,\ t \in \langle \alpha, \beta \rangle\}$.

Now we shall prove

THEOREM 2. *Suppose that hypotheses (i), (ii), (iii), (vi) are fulfilled, and let c (t) be a continuous function in T such that* $\bar{F}(a, t) = 0$. *Then equation* (1) *has in* Δ *a continuous solution depending on an arbitrary function. All these solutions fulfil the condition*

$$\varphi(a, t) = c(t) \quad \text{for} \quad t \in T.$$

P r o o f. We put

$$\psi(x, t) = \varphi(x, t) - c(t) \quad \text{for} \quad (x, t) \in \Delta. \tag{15}$$

If $\varphi(x, t)$ is a continuous solution of equation (1) in Δ fulfilling condition (14), then $\psi(x, t)$ is a continuous solution of the equation

$$\psi[f(x, t), t] = g(x, t)\, \psi(x, t) + \bar{F}(x, t) \tag{16}$$

such that $\psi(a, t) = 0$ for $t \in T$, and conversely. By induction we obtain from (16)

$$H_n(x, t) = \psi[f^n(x, t), t] - G_n(x, t)\, \psi(x, t) - \bar{F}[f^{n-1}(x, t), t]. \tag{17}$$

For every function $\psi_0(x, t)$ continuous in δ_0 and fulfilling the condition

$$\psi_0[f(x_0, t), t] = g(x_0, t) \cdot \psi_0(x_0, t) + \bar{F}(x_0, t),\ t \in T,$$

there exists a unique function $\psi(x, t)$ continuous and satisfying equation (16) in $(a, b)\times T$ and such that $\psi(x, t) = \psi_0(x, t)$ in δ_0[1]. We put $\psi(a, t) = 0$. It is enough to prove that for every $t_0 \in T$

$$\lim_{\substack{(x,t)\to(a,t_0)\\(x,t)\in\Delta}} \psi(x, t) = 0. \tag{18}$$

Let us fix an arbitrary $t_0 \in T$ and $<\alpha, \beta> \subset T$ such that $t_0 \in <\alpha, \beta>$. We put $L = \sup_{\delta} |\psi(x, t)|$. By (*vi*), the condition $F(a, t) = 0$ and the continuity of the function $F(x, t)$ in Δ, for every $\varepsilon > 0$ there exists an N such that

$$|H_n(x, t)| < \frac{\varepsilon}{3} \text{ in } \delta,\ n \geqslant N, \tag{19}$$

$$|G_n(x, t)| < \frac{\varepsilon}{3L} \text{ in } \delta,\ n \geqslant N, \tag{20}$$

$$|F(x, t)| < \frac{\varepsilon}{3} \text{ in } \{(x, t) : x \in <a, f^{N-1}(\bar{x}_0, t)>,\ t \in <\alpha, \beta>\}. \tag{21}$$

Let $(x, t) \in \delta = \{(x, t) : x \in (a, f^N(\bar{x}_0, t)),\ t \in <\alpha, \beta>\}$. There exist an $\bar{x} \in <f(\bar{x}_0, t), \bar{x}_0>$ and $n \geqslant N$ such that $x = f^n(\bar{x}, t)$. (17) gives then

$$\psi(x, t) = H_n(\bar{x}, t) + G_n(\bar{x}, t)\,\psi(\bar{x}, t) + \bar{F}[f^{n-1}(\bar{x}, t), t],$$

whence according to (19)—(21)

$$|\psi(x, t)| < \varepsilon \text{ in } \delta$$

which proves relation (18).

If φ_1 and φ_2 are continuous solutions of equation (1), then $\varphi(x, t) = \varphi_1(x, t) - \varphi_2(x, t)$ is a continuous solution of the equation

$$\varphi[f(x, t), x] = g(x, t)\,\varphi(x, t).$$

From [1] it follows that for every fixed $t \in T$ we have $\varphi(a, t) = 0$ and consequently $\varphi_1(a, t) = \varphi_2(a, t) = c(t)$, which completes the proof.

R e m a r k. Theorems 1 and 2 are also true for the equation

$$\varphi[f(x, t_1, \ldots, t_n)] = g(x, t_1, \ldots, t_n)\,\varphi(x) + F(x, t_1, \ldots, t_n).$$

---

[1]) The proof of this fact is analogous to the proof of theorem 1 in [3] and is therefore omitted.

REFERENCES

[1] B. Choczewski and M. Kuczma, *On the „indeterminate case" in the theory of a linear functional equation,* Fundamenta Mathematicae 58 (1966), 163—175.

[2] S. Czerwik: *On the continuous dependence of solutions of some functional equation on given functions Ann. Polon Math.* 24 (1971), 247—252.

[3] J. Kordylewski and M. Kuczma: *On some linear functional equation,* Ann. Polon. Math. 9 (1960), 119—136.

[4] J. Kordylewski and M. Kuczma: *On the continuous dependence of solutions of some functional equations on given functions I,* Ann. Polon. Math. 10 (1961), 41—48.

[5] J. Kordylewski and M. Kuczma: *On the continuous dependence of solutions of some functional equations on given functions II,* Ann. Polon. Math. 10 (1961), 167—174.

STEFAN CZERWIK

ZALEZNOSĆ OD PARAMETRU ROZWIĄZAŃ LINIOWEGO RÓWNANIA FUNKCYJNEGO

Streszczenie

W pracy dowodzi się twierdzenia 1 o istnieniu i jednoznaczności rozwiązań ciągłych równania (1) w zbiorze Δ w przypadku (C). Podaje się przykład dowodzący istotności założenia *(iv)*

Dowodzi się także twierdzenia 2 o istnieniu rozwiązań ciągłych w Δ w przypadku (B) zależnych od dowolnej funkcji.

Oddano do Redakcji 11. 3. 1971 r.