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Polynomiography for the Polynomial Infinity Norm via Kalantari's Formula and Nonstandard Iterations

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Abstract

In this paper, an iteration process, referred to in short as MMP, will be considered. This iteration is related to finding the maximum modulus of a complex polynomial over a unit disc on the complex plane creating intriguing images. Kalantari calls these images polynomiographs independently from whether they are generated by the root finding or maximum modulus finding process applied to any polynomial. We show that the images can be easily modified using different MMP methods (pseudo-Newton, MMP-Householder, methods from the MMP-Basic, MMP-Parametric Basic or MMP-Euler-Schröder Families of Iterations) with various kinds of non-standard iterations. Such images are interesting from three points of views: scientific, educational and artistic. We present the results of experiments showing automatically generated non-trivial images obtained for different modifications of root finding MMP-methods. The colouring by iteration reveals the dynamic behaviour of the used root finding process and its speed of convergence. The results of the present paper extend Kalantari's recent results in finding the maximum modulus of a complex polynomial based on Newton's process with the Picard iteration to other MMP-processes with various non-standard iterations.

Keywords: fractals, polynomiography, iterations, root finding, maximum modulus

1. Introduction

Kalantari defined polynomiography as the art and science of visualisation in approximation of the zeros of complex polynomials via fractal and non-fractal images created using the mathematical convergence properties of iteration functions [1, 2]. The well-known Newton method, as well as methods from the Basic Family and Euler-Schröder Family of Iterations will be used as iteration functions. The polynomiograph is a single two-dimensional image that presents the

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visualisation process of root finding for a given polynomial. Polynomiography, as a method of producing interesting graphics that could be widely used, was patented by Kalantari in the USA in 2005 [1].

In [3, 4], the authors presented a survey of some modifications of Kalantari's polynomiography based on the classic Newton's and the higher order Newton-like root finding methods for complex polynomials. Instead of the standard Picard's iteration, several different iteration processes were used. By combining different kinds of iterations, different convergence tests and different colouring methods they obtained a great variety of polynomiographs [4].

Recently, Kalantari presented the Maximum Modulus Principle (MMP) for complex polynomials and related it to the pseudo-Newton method together with some illustrative examples of polynomiographs [5]. The pseudo-Newton method produces intriguing images different from those obtained via the classic Newton's root finding process. In this paper, following [5], we explore further modifications of the algorithms for polynomiograph rendering obtained with the help of various iterations and root finding methods in their pseudo versions, which we call MMP methods. In comparison to [3, 4], we extend the list of iterations adding new iterations that have been presented in the literature recently. The actual list of iterations contains 18 items. Dependencies between iterations have been investigated and are presented on the diagram in Fig. 1.

The paper is organised as follows. Section 2 presents the Maximum Modulus Principle for polynomials and its connection with the pseudo-Newton method. Section 3 gives the definitions of the 18 types of iterations used in the fixed point theory and known from the literature. The following section, Section 4, describes selected root finding methods for a specific pseudo-polynomial. Section 5 is devoted to some modifications of the methods presented in Section 4. These modifications can be easily obtained using one of the non-standard iterations, instead of the Picard iteration. Section 6 describes the polynomiograph generation algorithm and Section 7 shows examples of polynomiographs. Finally, Section 8 concludes the paper and shows future directions on this subject.

2. Maximum Modulus Principle for Polynomials

Denote by p any non-constant complex polynomial on domain $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Next, state the following Maximum Modulus Problem: find a local maximum of $|p(z)|$ on D. It is known that, in this case, the Maximum Modulus Principle is satisfied [6] and states that

$$\|p\|_\infty = \max\{|p(z)| : z \in D\} \tag{1}$$

is attained at a boundary point of D. Further, a point $z_* \in D$ is a local maximum of $|p(z)|$ over D if and only if [5]

$$z_* = \left(\frac{p(z_*)}{p'(z_*)}\right) \Big/ \left(\left|\frac{p(z_*)}{p'(z_*)}\right|\right). \tag{2}$$

Formula (2) can be used to test if a given z is a local maximum of $|p(z)|$ on D. From (2) it follows that if z_* is a local maximum of $|p(z)|$ over D, then z_* is a fixed point of

$$F(z) = \left(\frac{p(z)}{p'(z)}\right) \Big/ \left(\left|\frac{p(z)}{p'(z)}\right|\right). \tag{3}$$

The fixed point of F can be found by the following iteration procedure:

$$z_{n+1} = F(z_n), \quad n = 0, 1, 2, \ldots, \tag{4}$$

where $z_0 \in \mathbb{C}$ is a given starting point.

Rather than solving $F(z) = z$, one can solve the following pseudo-polynomial equation [5]:

$$G(z) = p(z)|p'(z)| - zp'(z)|p(z)| = 0. \tag{5}$$

It is seen that $G(z_*) = 0$ if and only if either z_* is a fixed point of F, i.e. a local maximum of $|p(z)|$ over D, or z_* is a zero of $p(z)p'(z)$. This means that in general the number of zeros of $G(z) = 0$ is larger than the number of fixed points of F.

To solve (5), Kalantari proposed a method which he called the pseudo-Newton method [5]. This method takes the following form:

$$z_{n+1} = z_n - \frac{G_n(z_n)}{G'_n(z_n)}, \quad n = 0, 1, 2, \ldots, \tag{6}$$

where $z_0 \in \mathbb{C}$ is a given starting point and

$$G_n(z) = p(z)|p'(z_n)| - zp'(z)|p(z_n)|. \tag{7}$$

Observe that functions G_n and their derivatives G'_n with respect to z are changing from iteration to iteration and contain absolute value factors depending on z_n, which makes them constant values with respect to the z variable.

Introducing

$$N_n(z) = z - \frac{G_n(z)}{G'_n(z)} \tag{8}$$

we can express (6) in the following short form:

$$z_{n+1} = N_n(z_n), \quad n = 0, 1, 2, \ldots, \tag{9}$$

where $z_0 \in \mathbb{C}$ is a given starting point. This form uses the well-known Picard iteration.

3. Iterations

In fixed point theory there exist many theorems and methods that allow one to find fixed points of a given mapping. One of the techniques in the theory is an iterative approximation of the fixed points. We could use various kinds of iteration processes. Let us recall some of them.

Let $T : X \to X$ be a mapping on a metric space (X, d), where d is a metric. Further, let $x_0 \in X$ be a starting point.

- Picard iteration (1890) [7]:

$$x_{n+1} = T(x_n), \quad n = 0, 1, 2, \dots, \tag{10}$$

- Mann iteration (1953) [8]:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n), \quad n = 0, 1, 2, \dots, \tag{11}$$

where $\alpha_n \in (0, 1]$ for all $n \in \mathbb{N}$.

- Ishikawa iteration (1974) [9]:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n T(x_n), \quad n = 0, 1, 2, \dots, \end{cases} \tag{12}$$

where $\alpha_n \in (0, 1]$ and $\beta_n \in [0, 1]$ for all $n \in \mathbb{N}$.

- Noor iteration (2000) [10]:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n T(z_n), \\ z_n = (1 - \gamma_n)x_n + \gamma_n T(x_n), \quad n = 0, 1, 2, \dots, \end{cases} \tag{13}$$

where $\alpha_n \in (0, 1]$ and $\beta_n, \gamma_n \in [0, 1]$ for all $n \in \mathbb{N}$.

- Suantai iteration (2005) [11]:

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n T(y_n) + \beta_n T(z_n), \\ y_n = (1 - a_n - b_n)x_n + a_n T(z_n) + b_n T(x_n), \\ z_n = (1 - \gamma_n)x_n + \gamma_n T(x_n), \quad n = 0, 1, 2, \dots, \end{cases} \tag{14}$$

where $\alpha_n, \beta_n, \gamma_n, a_n, b_n \in [0, 1]$, $\alpha_n + \beta_n \in [0, 1]$, $a_n + b_n \in [0, 1]$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$.

- S-iteration (2007) [12]:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)T(x_n) + \alpha_n T(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n T(x_n), \quad n = 0, 1, 2, \dots, \end{cases} \tag{15}$$

where $\alpha_n \in (0, 1]$ and $\beta_n \in [0, 1]$ for all $n \in \mathbb{N}$.

- SP iteration (2011) [13]:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)y_n + \alpha_n T(y_n), \\ y_n = (1 - \beta_n)z_n + \beta_n T(z_n), \\ z_n = (1 - \gamma_n)x_n + \gamma_n T(x_n), \quad n = 0, 1, 2, \dots, \end{cases} \tag{16}$$

where $\alpha_n \in (0, 1]$ and $\beta_n, \gamma_n \in [0, 1]$ for all $n \in \mathbb{N}$.

- CR iteration (2012) [14]:

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n + \alpha_n T(y_n), \\ y_n = (1-\beta_n)T(x_n) + \beta_n T(z_n), \\ z_n = (1-\gamma_n)x_n + \gamma_n T(x_n), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{17}$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

- Khan iteration (2013) [15]:

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = (1-\alpha_n)x_n + \alpha_n T(x_n), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{18}$$

where $\alpha_n \in (0,1]$ for all $n \in \mathbb{N}$.

- Karakaya iteration (2013) [16]:

$$\begin{cases} x_{n+1} = (1-\alpha_n-\beta_n)y_n + \alpha_n T(y_n) + \beta_n T(z_n), \\ y_n = (1-a_n-b_n)z_n + a_n T(z_n) + b_n T(x_n), \\ z_n = (1-\gamma_n)x_n + \gamma_n T(x_n), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{19}$$

where $\alpha_n, \beta_n, \gamma_n, a_n, b_n \in [0,1]$, $\alpha_n + \beta_n \in [0,1]$, $a_n + b_n \in [0,1]$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$.

- S*-iteration (2013) [17]:

$$\begin{cases} x_{n+1} = (1-\alpha_n)T(x_n) + \alpha_n T(y_n), \\ y_n = (1-\beta_n)T(x_n) + \beta_n T(z_n), \\ z_n = (1-\gamma_n)x_n + \gamma_n T(x_n), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{20}$$

where $\alpha_n \in (0,1]$ and $\beta_n, \gamma_n \in [0,1]$ for all $n \in \mathbb{N}$.

- Picard-S iteration (2014) [18]:

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = (1-\alpha_n)T(x_n) + \alpha_n T(z_n), \\ z_n = (1-\beta_n)x_n + \beta_n T(x_n), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{21}$$

where $\alpha_n \in (0,1]$ and $\beta_n \in [0,1]$ for all $n \in \mathbb{N}$.

- Thakur iteration (2014) [19]:

$$\begin{cases} x_{n+1} = (1-\alpha_n)T(x_n) + \alpha_n T(y_n), \\ y_n = (1-\beta_n)z_n + \beta_n T(z_n), \\ z_n = (1-\gamma_n)x_n + \gamma_n T(x_n), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{22}$$

where $\alpha_n \in (0,1]$ and $\beta_n, \gamma_n \in [0,1]$ for all $n \in \mathbb{N}$.

- Abbas iteration (2014) [20]:

$$\begin{cases} x_{n+1} = (1-\alpha_n)T(y_n) + \alpha_n T(z_n), \\ y_n = (1-\beta_n)T(x_n) + \beta_n T(z_n), \\ z_n = (1-\gamma_n)x_n + \gamma_n T(x_n), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{23}$$

  where $\alpha_n \in (0, 1]$ and $\beta_n, \gamma_n \in [0, 1]$ for all $n \in \mathbb{N}$.

- Karakaya iteration (2015) [21]:

$$\begin{cases} x_{n+1} = T((1-\alpha_n)y_n + \alpha_n T(y_n)), \\ y_n = T((1-\beta_n)x_n + \beta_n T(x_n)), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{24}$$

  where $\alpha_n \in (0, 1]$, $\beta_n \in [0, 1]$ for all $n \in \mathbb{N}$.

- P-iteration (2015) [22]:

$$\begin{cases} x_{n+1} = (1-\alpha_n)T(z_n) + \alpha_n T(y_n), \\ y_n = (1-\beta_n)z_n + \beta_n T(z_n), \\ z_n = (1-\gamma_n)x_n + \gamma_n T(x_n), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{25}$$

  where $\alpha_n \in (0, 1]$ and $\beta_n, \gamma_n \in [0, 1]$ for all $n \in \mathbb{N}$.

- Sintunavarat iteration (2016) [23]:

$$\begin{cases} x_{n+1} = (1-\alpha_n)T(y_n) + \alpha_n T(z_n), \\ y_n = (1-\beta_n)x_n + \beta_n z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T(x_n), \quad n = 0, 1, 2, \ldots \end{cases} \tag{26}$$

  where $\alpha_n \in (0, 1]$ and $\beta_n, \gamma_n \in [0, 1]$ for all $n \in \mathbb{N}$.

- Thakur iteration (2016) [24]:

$$\begin{cases} x_{n+1} = T(y_n), \\ y_n = T((1-\alpha_n)x_n + \alpha_n z_n), \\ z_n = (1-\beta_n)x_n + \beta_n T(x_n), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{27}$$

  where $\alpha_n \in (0, 1]$, $\beta_n \in [0, 1]$ for all $n \in \mathbb{N}$.

The presented iterations for particular values of the parameters can be reduced to other iterations. For instance, it is easily seen that the Ishikawa iteration with $\beta_n = 0$ for all $n \in \mathbb{N}$ is a Mann iteration, and when $\beta_n = 0$, $\alpha_n = 1$ for all $n \in \mathbb{N}$ is the Picard iteration. The dependencies between all the mentioned iterations are shown in Fig. 1.

In the rest of the paper we will work in the Banach space $X = \mathbb{C}$ with the standard norm. We take $z_0 \in \mathbb{C}$ and $\alpha_n = \alpha$, $\beta_n = \beta$, $\gamma_n = \gamma$, $a_n = a$, $b_n = b$ for all $n \in \mathbb{N}$, such that $\alpha \in (0, 1], \beta, \gamma, a, b \in [0, 1]$, $\alpha + \beta \in (0, 1]$ and $a + b \in [0, 1]$. Naturally, if $\alpha + \beta \in (0, 1]$, then $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \sum_{n=0}^{\infty}(\alpha + \beta) = \infty$.

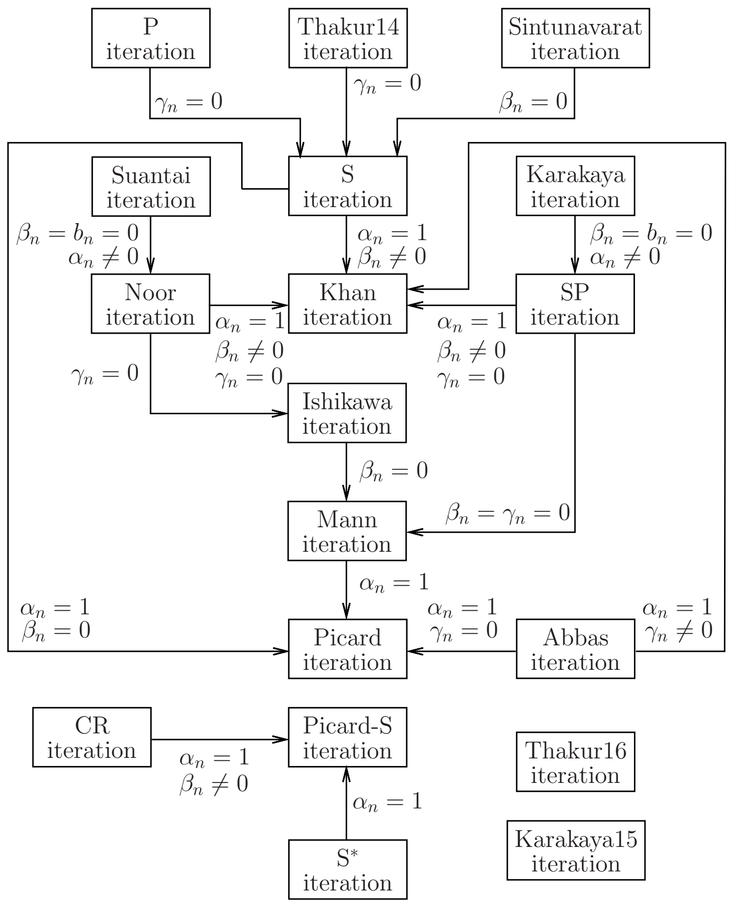


Figure 1: The diagram of iterations' dependencies.

4. Root Finding Methods for G_n

In the literature, many root finding methods are known, e.g. the Traub-Ostrowski method [25], the Householder method [26], the Harmonic Mean Newton's method [27], and higher order methods from the Basic Family or Euler-Schröder's Family of Iterations [2], to mention but a few. Every such method can be easily adopted for finding roots of G_n. To adopt any root finding method we only need to use formally functions G_n and their derivatives in formulas defining any particular method. This approach is justified because any k–th derivative of G_n is well-defined, as is shown in the following theorem.

Theorem 1. *Let G_n be given by* (7)*. Then, for a fixed n function G_n is C^∞ and for every $k \geq 1$*

$$G_n^{(k)}(z) = p^{(k)}(z)(|p'(z_n)| - k|p(z_n)|) - zp^{(k+1)}(z)|p(z_n)|. \tag{28}$$

Proof. For a fixed n, the values $|p'(z_n)|$, $|p(z_n)|$ are constant. Therefore, the terms $p(z)|p'(z_n)|$, $zp'(z)|p(z_n)|$ are polynomials of argument z. Then, G_n, as the difference of polynomials, is also a polynomial and so it is in C^∞.

Now, we prove (28) by induction. For $k = 1$ we have

$$\begin{aligned} G_n'(z) &= p'(z)|p'(z_n)| - |p(z_n)|(p'(z) + zp''(z)) \\ &= p'(z)(|p'(z_n)| - |p(z_n)|) - zp''(z)|p(z_n)|. \end{aligned}$$

Assume that the statement is true for some $k \geq 1$, i.e.

$$G_n^{(k)}(z) = p^{(k)}(z)(|p'(z_n)| - k|p(z_n)|) - zp^{(k+1)}(z)|p(z_n)|. \tag{29}$$

For $k+1$, we have $G_n^{(k+1)}(z) = (G_n^{(k)}(z))'$. From the inductive hypothesis (29) we get

$$\begin{aligned} G_n^{(k+1)}(z) &= (p^{(k)}(z)(|p'(z_n)| - k|p(z_n)|) - zp^{(k+1)}(z)|p(z_n)|)' \\ &= p^{(k+1)}(z)(|p'(z_n)| - k|p(z_n)|) - |p(z_n)|(p^{(k+1)}(z) + zp^{(k+2)}(z)) \\ &= p^{(k+1)}(z)(|p'(z_n)| - (k+1)|p(z_n)|) - zp^{(k+2)}(z)|p(z_n)| \end{aligned}$$

Therefore, (28) follows by induction for all $k \geq 1$. □

Next, we present some selected formulas for root finding of G_n, which we will call MMP-methods. We start with the MMP-Householder method:

$$z_{n+1} = N_n(z_n) - \frac{G_n(z_n)^2 G_n''(z_n)}{2G_n'(z_n)^3}, \quad n = 0, 1, 2, \ldots, \tag{30}$$

where N_n is the pseudo-Newton method given by (8).

For the Basic Family of Iterations presented by Kalantari in [2], we can introduce its MMP version in the following way. Let p be a complex polynomial with $\deg p \geq 2$ and G_n functions associated with p. For any n we define a sequence of functions $D_{m,n} : \mathbb{C} \to \mathbb{C}$ as follows: $D_{0,n}(z) = 1$ and for $m > 0$ let

$$D_{m,n}(z) = \det \begin{bmatrix} G_n'(z) & \frac{G_n''(z)}{2!} & \dots & \frac{G_n^{(m-1)}(z)}{(m-1)!} & \frac{G_n^{(m)}(z)}{m!} \\ G_n(z) & G_n'(z) & \ddots & \ddots & \frac{G_n^{(m-1)}(z)}{(m-1)!} \\ 0 & G_n(z) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{G_n''(z)}{2!} \\ 0 & 0 & \dots & G_n(z) & G_n'(z) \end{bmatrix}. \tag{31}$$

The elements of the MMP-Basic Family of Iterations are then defined as:

$$B_{m,n}(z) = z - G_n(z)\frac{D_{m-2,n}(z)}{D_{m-1,n}(z)}, \quad m = 2, 3, \dots. \tag{32}$$

Let us see how the first three elements of the MMP-Basic Family look like:

$$B_{2,n}(z) = z - \frac{G_n(z)}{G_n'(z)}, \tag{33}$$

$$B_{3,n}(z) = z - \frac{2G_n'(z)G_n(z)}{2G_n'(z)^2 - G_n''(z)G_n(z)}, \tag{34}$$

$$B_{4,n}(z) = z - \frac{6G_n'(z)^2G_n(z) - 3G_n''(z)G_n(z)^2}{G_n'''(z)G_n(z)^2 + 6G_n'(z)^3 - 6G_n''(z)G_n'(z)G_n(z)}. \tag{35}$$

One can easily see that $B_{2,n}$ is the pseudo-Newton's method, whereas $B_{3,n}$ is the MMP-Halley's method.

By using functions $D_{m,n}$ following [2], one can define the MMP-Parametric Basic Family of Iterations:

$$B_{m,n,\lambda}(z) = z - \lambda G_n(z)\frac{D_{m-2,n}(z)}{D_{m-1,n}(z)}, \tag{36}$$

where $m = 2, 3, \dots$ and $\lambda \in \mathbb{C}$. Let us note that for $\lambda = 1$ the MMP-Parametric Basic Family of Iterations reduces to the MMP-Basic Family of Iterations.

In [2], we can find another family of iterations, namely the Euler-Schröder Family. The initial elements of the MMP version of this family have the following form:

$$E_{2,n}(z) = z - \frac{G_n(z)}{G_n'(z)}, \tag{37}$$

$$E_{3,n}(z) = E_{2,n}(z) + \left(\frac{G_n(z)}{G_n'(z)}\right)^2 \frac{G_n''(z)}{2G_n'(z)}, \tag{38}$$

$$E_{4,n}(z) = E_{3,n}(z) - \left(\frac{G_n(z)}{G_n'(z)}\right)^3 \left(\frac{G_n'''(z)}{6G_n'(z)} - \frac{G_n''(z)}{2G_n'(z)^2}\right), \tag{39}$$

$$E_{5,n}(z) = E_{4,n}(z) + \left(\frac{G_n(z)}{G_n'(z)}\right)^4 \left(\frac{G_n^{IV}(z)}{4!G_n'(z)} - \frac{5G_n''(z)G_n'''(z)}{12G_n'(z)^2} + \frac{5G_n''(z)^3}{8G_n'(z)^3}\right). \tag{40}$$

One can easily see that $E_{2,n}$ is the pseudo-Newton's method. The construction of the other elements of the family can be made following [2].

5. Modifications of Root-Finding Methods for G_n

In [4], the authors have presented some modifications of the polynomiograph's generation process. Following the ideas from the article we can introduce similar modifications to the MMP versions of the root finding methods. Let us denote by R_n any root finding method in its MMP version, e.g. those from Section 4. Now, let us replace the standard Picard iteration in the MMP versions of the root finding methods by one of the non-standard iterations described in Section 3, in which R_n plays the role of T. Moreover, following [4], we can replace the real parameters of those iterations with the complex ones.

Because the modifications are straightforward, we present only examples for selected iterations ($\Re(z)$ denotes the real part of z):

- P-iteration:

$$\begin{cases} z_{n+1} = (1-\alpha)R_n(w_n) + \alpha R_n(v_n), \\ v_n = (1-\beta)w_n + \beta R_n(w_n), \\ w_n = (1-\gamma)z_n + \gamma R_n(z_n), \quad n = 0, 1, 2, \dots, \end{cases} \tag{41}$$

  where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\Re(\alpha) \in (0, 1]$, $\Re(\beta), \Re(\gamma) \in [0, 1]$,

- CR iteration:

$$\begin{cases} z_{n+1} = (1-\alpha)v_n + \alpha R_n(v_n), \\ v_n = (1-\beta)R_n(z_n) + \beta R_n(w_n), \\ w_n = (1-\gamma)z_n + \gamma R_n(z_n), \quad n = 0, 1, 2, \dots, \end{cases} \tag{42}$$

  where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\Re(\alpha), \Re(\beta), \Re(\gamma) \in [0, 1]$,

- Karakaya iteration (2015):

$$\begin{cases} z_{n+1} = R_n((1-\alpha)v_n + \alpha R_n(v_n)), \\ v_n = R_n((1-\beta)z_n + \beta R_n(z_n)), \quad n = 0, 1, 2, \dots, \end{cases} \tag{43}$$

  where $\alpha, \beta \in \mathbb{C}$ and $\Re(\alpha) \in (0, 1]$, $\Re(\beta) \in [0, 1]$.

6. Polynomiograph Generation

The generation algorithm of polynomiograph for MMP-methods is very similar to the algorithm used for the standard root-finding methods [4]. First, we take a polynomial $p \in \mathbb{C}[Z]$. Then, for each starting point z_0 in the area $A \subset \mathbb{C}$ we choose the iteration process from Section 3 and a root-finding method in its MMP version, as described in Section 5. The iteration proceeds till the convergence criterion (test) is satisfied or the maximum number of iterations is reached. The standard convergence test has the following form:

$$|z_{n+1} - z_n| < \varepsilon, \tag{44}$$

where $\varepsilon > 0$. In [28], we can find other convergence tests that are based on metric and non-metric conditions, e.g.

$$|0.01(z_{i+1} - z_i)| + |0.029|z_{i+1}|^2 - 0.03|z_i|^2| < \varepsilon, \tag{45}$$

$$|0.04\Re(z_{i+1} - z_i)| < \varepsilon \vee |0.05\Im(z_{i+1} - z_i)| < \varepsilon, \tag{46}$$

$$|0.4\Re(z_{i+1} - z_i)|^2 < \varepsilon \wedge |\Im(z_{i+1} - z_i)|^2 < \varepsilon. \tag{47}$$

When the iteration process ends, i.e. the maximum number K of iterations is reached or the convergence test is satisfied, we assign a colour to the starting point z_0 based on the number of performed iterations and a chosen colour map.

The pseudocode of the algorithm is presented in Algorithm 1. In the algorithm, the selected iteration process from Section 3 is denoted as I_q. The index q is a vector of parameters of the iteration method, i.e. $q \in \mathbb{C}^N$, where N is the number of parameters of the iteration. The convergence test is denoted as T_t, where $t \in \mathbb{R}^M$ is a vector of parameters of the test. Moreover, the colour map is represented as a table of C colours.

Algorithm 1: Polynomiograph generation

Input: $p \in \mathbb{C}[Z]$ – polynomial, $A \subset \mathbb{C}$ – area, K – the maximum number of iterations, I_q – iteration method, $q \in \mathbb{C}^N$ – parameters of the iteration I_q, R_n – root-finding method in its pseudo version, T_t – convergence test, $t \in \mathbb{R}^M$ – parameters of the convergence test T_t, $colourmap[0..C-1]$ – colour map with C colours.

Output: Polynomiograph for the area A.

for $z_0 \in A$ **do**
    $n = 0$
    while $n < K$ **do**
        $z_{n+1} = I_q(R_n, p, z_n)$
        if $T_t(z_n, z_{n+1}) = true$ **then**
            break
        $n = n + 1$
    $i = \lfloor (C-1)\frac{n}{K} \rfloor$
    colour z_0 with $colourmap[i]$

7. Examples of Polynomiographs

In this section, some examples of the polynomiographs obtained by using the methods described in the previous sections are presented. In the first example we present the use of different iterations in Kalantari's pseudo-Newton method. The common parameters used in the example were the following: $p(z) = z^3 - 1$, $A = [-3, 3]^2$, $K = 40$, convergence test (44) with $\varepsilon = 0.001$, and the iterations' parameters were as follows:

- Mann iteration: $\alpha = 0.4 + 0.1\mathbf{i}$,
- Ishikawa iteration: $\alpha = 0.7$, $\beta = 0.6$,
- Noor iteration: $\alpha = 0.7 + 0.4\mathbf{i}$, $\beta = 0.77 + 0.26\mathbf{i}$, $\gamma = 0.19 + 0.21\mathbf{i}$,
- Suantai iteration: $\alpha = 0.2$, $\beta = 0.2$, $\gamma = 1.0$, $a = 0.1$, $b = 0.1$,
- S iteration: $\alpha = 0.95 - 0.5\mathbf{i}$, $\beta = 0.5 + 0.5\mathbf{i}$,
- SP iteration: $\alpha = 0.7$, $\beta = 0.85$, $\gamma = 0.5$,
- CR iteration: $\alpha = 0.9$, $\beta = 0.9$, $\gamma = 0.9$,
- Khan iteration: $\alpha = 0.3$,
- Karakaya iteration (2013): $\alpha = 0.2$, $\beta = 0.2$, $\gamma = 0.05$, $a = 0.1$, $b = 0.1$,
- S* iteration: $\alpha = 0.5 + \mathbf{i}$, $\beta = 0.5 + \mathbf{i}$, $\gamma = 0.3 + \mathbf{i}$,
- Picard-S iteration: $\alpha = 0.5 - 0.7\mathbf{i}$, $\beta = 0.5 - 0.7\mathbf{i}$,
- Thakur iteration (2014): $\alpha = 0.7$, $\beta = 0.35$, $\gamma = 0.9$,
- Abbas iteration: $\alpha = 0.05 + 0.9\mathbf{i}$, $\beta = 0.95$, $\gamma = 0.05$,
- Karakaya iteration (2015): $\alpha = 0.95 - \mathbf{i}$, $\beta = 0.05 - \mathbf{i}$,
- P iteration: $\alpha = 0.5$, $\beta = 0.9$, $\gamma = 0.05$,
- Sintunavarat iteration: $\alpha = 0.94$, $\beta = 0.25$, $\gamma = 0.75$,
- Thakur iteration (2016): $\alpha = 0.9$, $\beta = 0.7$.

Figs. 2, 3 present the obtained polynomiographs. From the images, we see that changing the iteration method alters the shape of the pattern compared to the pattern obtained for the standard Picard iteration. Moreover, we can observe that the use of complex parameters adds swirls and twists to the obtained patterns. This makes the images look more dynamic and vivid.

In the example with the pseudo-Newton method, for each iteration we used only one set of parameters' values. In our study, we tried other values for the parameters and found the images in general to be non-trivial and attractive. As an example, in Fig. 4 we present polynomiographs for the Mann iteration with different values of the parameter α: (a) $0.2 + 0.1\mathbf{i}$, (b) $0.4 + 0.1\mathbf{i}$, (c) $0.6 + 0.1\mathbf{i}$, (d) $0.8 + 0.1\mathbf{i}$. It is seen that the images in Fig. 4 change "smoothly" with a "small" change of parameter α. We observed similar effects for the other types of iteration processes.

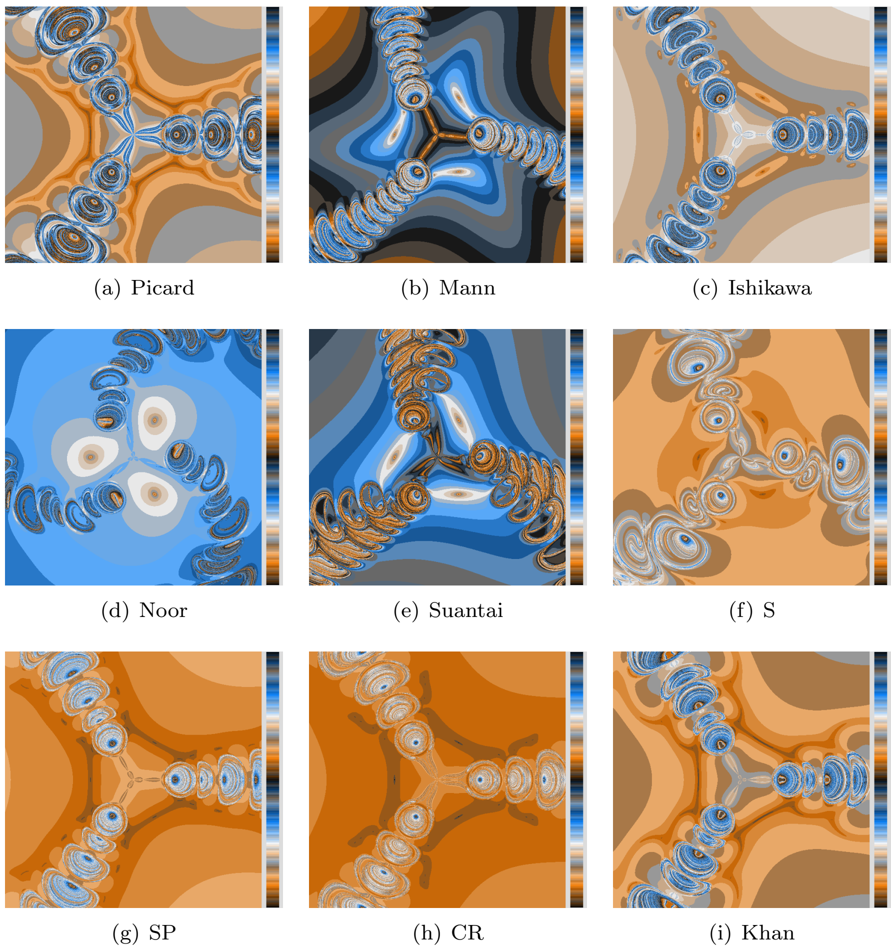
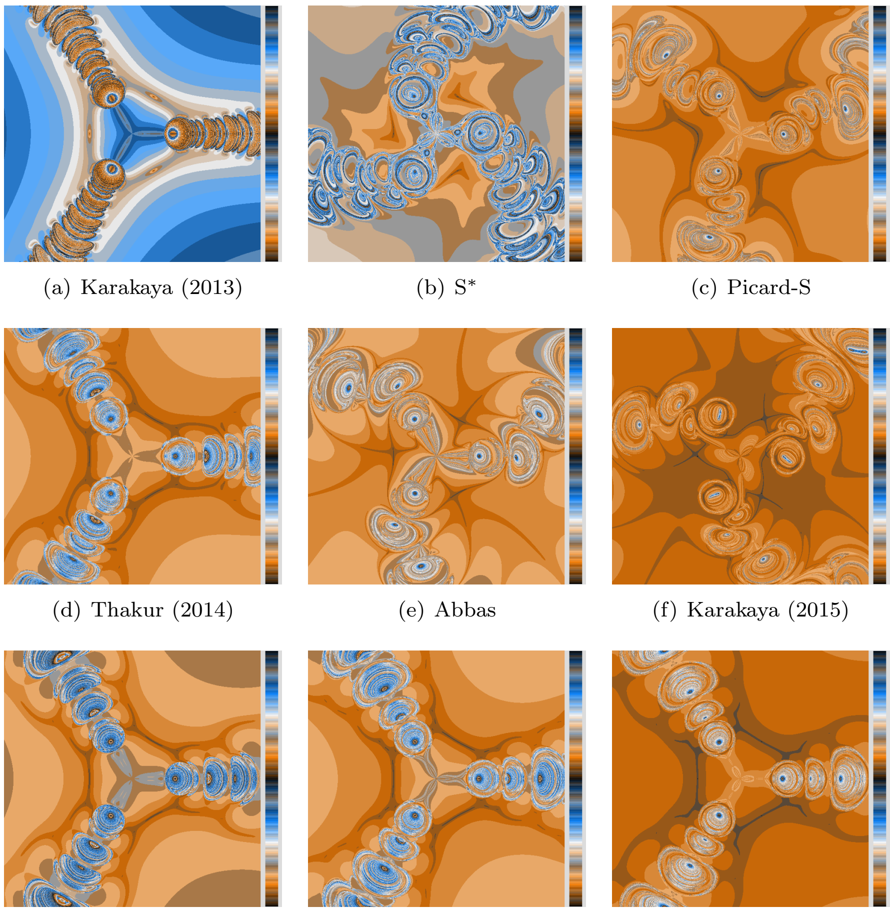
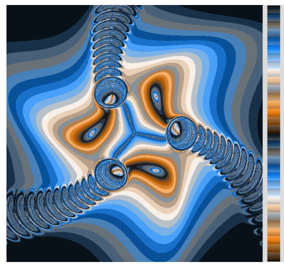


Figure 2: Examples of polynomiographs for the pseudo-Newton method and various iterations.

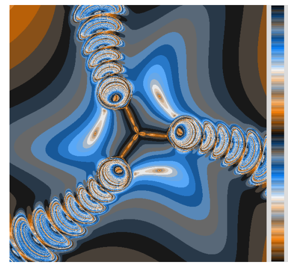


(a) Karakaya (2013)
(b) S*
(c) Picard-S
(d) Thakur (2014)
(e) Abbas
(f) Karakaya (2015)
(g) P
(h) Sintunavarat
(i) Thakur (2016)

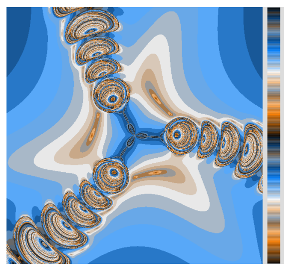
Figure 3: Examples of polynomiographs for the pseudo-Newton method and various iterations (cont.).



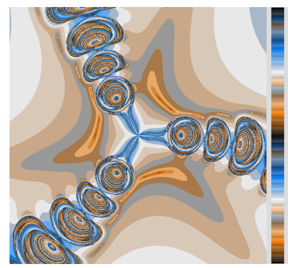
(a) $\alpha = 0.2 + 0.1\mathbf{i}$



(b) $\alpha = 0.4 + 0.1\mathbf{i}$



(c) $\alpha = 0.6 + 0.1\mathbf{i}$

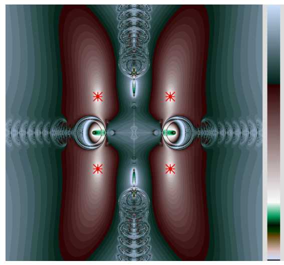


(d) $\alpha = 0.8 + 0.1\mathbf{i}$

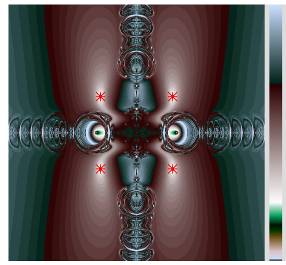
Figure 4: Examples of polynomiographs for various values of the α parameter in the Mann iteration used for the pseudo-Newton method

In the second example, we present the use of different MMP-methods for G_n. The common parameters used to generate the images in this example were the following: $p(z) = z^4 + z^2 - 1$, $A = [-2.5, 2.5]^2$, $K = 50$, convergence test (44) with $\varepsilon = 0.001$. The maxima of $|p(z)|$ are attained for $0.707107 + 0.707107\mathbf{i}$, $-0.707107+0.707107\mathbf{i}$, $-0.707107-0.707107\mathbf{i}$, $0.707107-0.707107\mathbf{i}$. In Figs. 5–7 the points with the maxima are marked with a red asterisk. Fig. 5 shows images obtained using the CR iteration with $\alpha = 0.1$, $\beta = 0.1$, $\gamma = 0.85$ and following MMP-methods: (a) pseudo-Newton, (b) MMP-Householder, (c) MMP-Halley, (d) MMP-E_3. Images obtained using the same MMP-methods but with the use of the Noor iteration with $\alpha = 0.7$, $\beta = 0.5$, $\gamma = 0.5$ are presented in Fig. 6. Finally, in Fig. 7 we see images obtained with the Thakur iteration (2016) with $\alpha = 0.5$, $\beta = 0.5$. Comparing images in each figure we can observe that the use of different MMP-methods has a great impact on the shape of the pattern. Moreover, we see further examples of using different iteration methods with a fixed MMP-method (compare corresponding images in each of the figures).

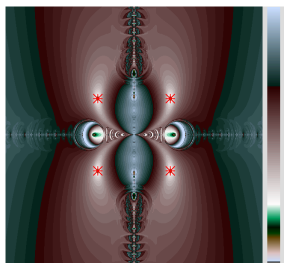
Furthermore, polynomiographs from Figs. 5–7 present the dynamic behaviour of various MMP-processes. Colours and their gradients show how many iterations are needed to find $\max |p(x)|$ and the speed of convergence for a particular MMP-process starting from a given point on the polynomiograph respectively. The number of iterations uniformly increases along the vertical colour bar palette (the bottom and top corresponds to 0 and 50 iterations respectively). Looking at the colours and shapes of the polynomiographs one can see that the



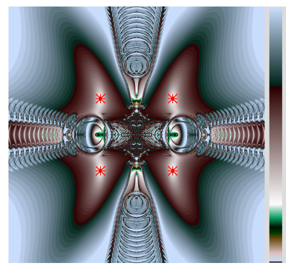
(a) pseudo-Newton



(b) MMP-Householder

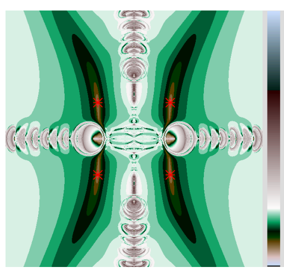


(c) MMP-Halley

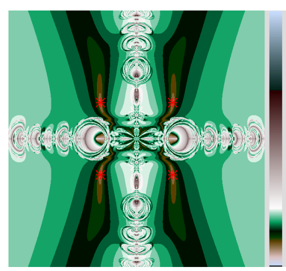


(d) MMP-E_3

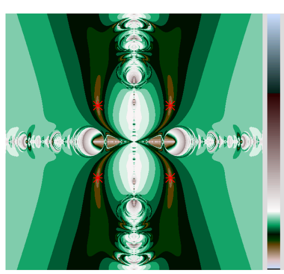
Figure 5: Examples of polynomiographs for various MMP-methods and the CR iteration.



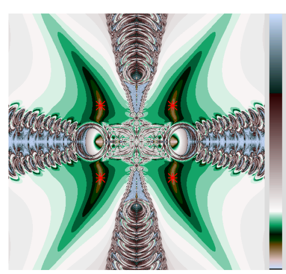
(a) pseudo-Newton



(b) MMP-Householder



(c) MMP-Halley



(d) MPP-E_3

Figure 6: Examples of polynomiographs for various MMP-methods and the Noor iteration.

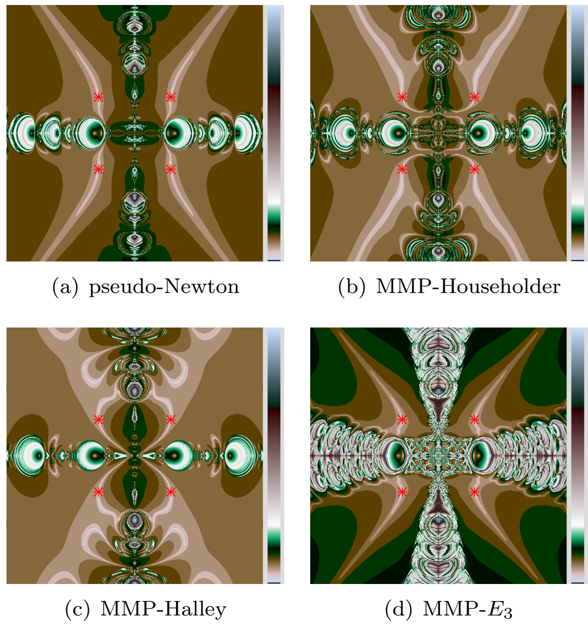


Figure 7: Examples of polynomiographs for various MMP-methods and the Thakur iteration (2016).

use of different iteration processes and MMP-methods changes the speed of convergence of the root-finding process for G_n – for some points the convergence is faster and for others it is slower. The speed depends on the MMP-method, the iteration and the values of the parameters used.

The last example, presented in Fig. 8, shows various patterns obtained using the proposed methods and modifications. The parameters used to generate these images were as follows:

(a) $p(z) = z^4 + z^2 - 1$, $A = [-2.5, 2.5]^2$, $K = 50$, Thakur iteration (2016) with $\alpha = 0.5$, $\beta = 0.5$, pseudo-Newton method, convergence test (46) with $\varepsilon = 0.001$,

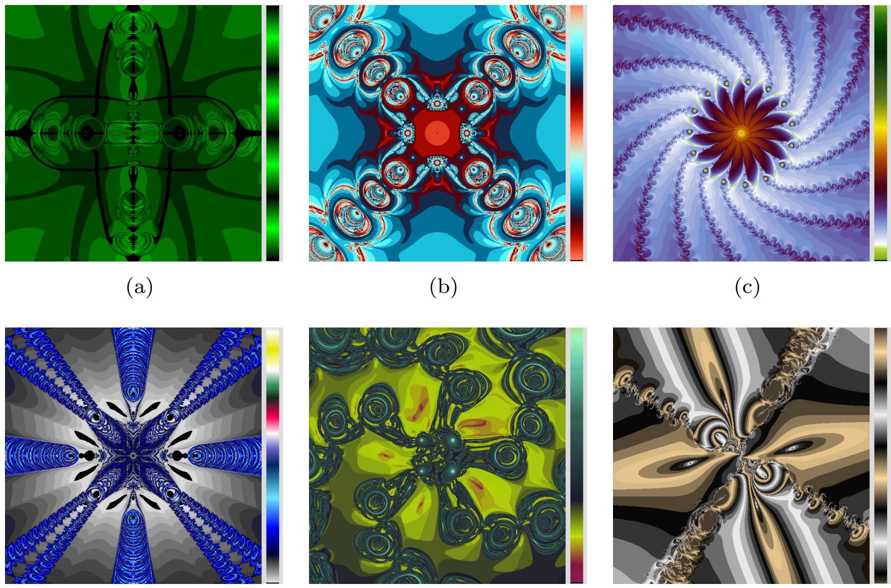
(b) $p(z) = z^5 + z$, $A = [-2.5, 2.5]^2$, $K = 20$, Picard iteration, MMP-Householder method, convergence test (44) with $\varepsilon = 0.001$,

(c) $p(z) = z^{15} + 1$, $A = [-2.5, 2.5]^2$, $K = 100$, Ishikawa iteration with $\alpha = 0.65 - 0.35\mathbf{i}$, $\beta = 0.5 - 0.35\mathbf{i}$, MMP-Halley method, convergence test (44) with $\varepsilon = 0.001$,

(d) $p(z) = z^8 - z^4 - 2$, $A = [-2.5, 2.5]^2$, $K = 100$, Khan iteration with $\alpha = 0.5$, MMP-E_3 method, convergence test (47) with $\varepsilon = 0.001$,

(e) $p(z) = z^5 + z - 5$, $A = [-2.5, 2.5]^2$, $K = 50$, Mann iteration with $\alpha = 0.95 - \mathbf{i}$, MMP-Householder method, convergence test (44) with $\varepsilon = 0.001$,

(f) $p(z) = (4 + 4\mathbf{i})z^4 + 8\mathbf{i}z^2 + 4$, $A = [-2.5, 2.5]^2$, $K = 30$, CR iteration with $\alpha = 0.35$, $\beta = 0.1$, $\gamma = 0.85$, MMP-Halley method, convergence test (44) with $\varepsilon = 0.001$.



(a) (b) (c)

(d) (e) (f)

Figure 8: Examples of polynomiographs for various parameters.

From the images, we see that using different combinations of the parameters, e.g. MMP-methods, iteration processes etc., we are able to obtain very diverse and interesting fractal patterns that have potential artistic applications.

8. Conclusions and Future Work

Polynomiographs are good examples of the connection between mathematics and art. They also reveal hidden information on the character and speed of convergence to polynomials' or pseudo-polynomials' roots. In this paper, we presented some modifications of Kalantari's results on the visualisation of the root-finding process for the pseudo-polynomial G. We showed that the Maximum Modulus Principle for complex polynomials is enriched by different types of iterations, MMP-methods, convergence tests and different colour maps that can serve as a good source of impressive and intriguing images. Apart from being artistically valuable, the images reveal information about convergence and its speed for traditional root-finding iterations, as well as MMP-iterations that converge to a maximum of the polynomial's modulus.

It is known that the shape of polynomiographs can be easily modified in a predictable way by changing the roots of polynomial p. It is interesting to question whether the same would occur for polynomiographs obtained with the MMP-methods. The performed experiments show that the answer concerning the "continuity" of images with respect to "small" changes of the roots of p is affirmative. Moreover, the results of the paper can be further extended by using the multipoint methods [29, 30].

Recently, in [31], a study on the use of various switching processes in polynomiography was presented. We can investigate similar switching processes and introduce new ones with the MMP-methods. Another interesting direction of future studies could rely on replacing complex numbers by more general ones, e.g. dual and double numbers as used in [32] for defining the q-system fractals.

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