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Pseudoinversion Fractals

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Abstract. In this paper, we present some modifications of inversion fractals. The first modification is based on the use of different metrics in the inversion transformation. Moreover, we propose a switching process between different metric spaces. All the proposed modifications allowed us to obtain new and diverse fractal patterns that differ from the original inversion fractals.

Keywords: fractal, pseudoinversion, computer art

1 Introduction

Fractals discovered by Mandelbrot in 1970s are used to model complex shapes such as clouds, plants, mountains, sea-shores. They are also applied in the field of art and computer graphics. Many different methods of obtaining fractal patterns were proposed in the literature, e.g., dynamical systems [4], hyperbolic geometry [6], complex numbers [5] or iterated function systems [9]. One of the recent methods is the use of inversion transformation of the star-shaped sets [2, 3]. This type of fractals are called inversion fractals. In this paper we propose some modifications of the inversion transformation that lead to new fractal patterns.

The paper is organized as follows. In Sec. 2, we briefly introduce the inversion fractals and the algorithm to generate them. Next, in Sec. 3, we introduce some modifications of inversion fractals. The first modification is based on the use of pair of metrics in the inversion transformation and the second modification uses switching process between pairs of metrics. Some examples of fractal patterns obtained with the proposed modifications are presented in Sec. 4. Finally, in Sec. 5, we give some concluding remarks.

2 Inversion Fractals

To introduce the psuedoinversion fractals firstly we must know what the inversion fractals are. The first fractals of this type appeared about 2000 in [1]. They were based on circle inversion. Later in [2] a generalization from circles to the star-shaped sets was introduced. Some further generalizations, namely the use of iteration process from fixed-point theory and the use of q-systems, were presented in [3].

Following [3] let us start with some definitions.

Definition 1. *A set S in a metric space $(\mathbb{R}^2, d_e)$, where d_e is the Euclidean distance, is* star-shaped *if there exists a point $z \in int\, S$ (int S means the interior of S) such that for all points $p \in S$ the line segment $\overline{zp}$ lies entirely within S. The locus of the points z having the above property is the* kernel *of S and is denoted by* $\ker S$.

Let us assume that we have a star-shaped set S, some point $o \in \ker S$ and point $p \neq o$ for which we want to calculate the inversion. We start by shooting a ray r from o in the direction $p - o$, i.e., $r(t) = o + t(p - o)$, where $t \in [0, \infty)$. Then, we find the intersection point b of r and the boundary of S.

Definition 2. *Point p' is said to be the* inverse *of p with respect to S if it satisfies the following equation:*

$$d_e(o, p) \cdot d_e(o, p') = [d_e(o, b)]^2. \tag{1}$$

Point o is called the centre of inversion. *The transformation that takes p and transforms it into p' is called the* star-shaped set inversion transformation *and it is denoted by I_S.*

The inversion transformation can be extended also to o in a following way: $I_S(o) = \infty$ and $I_S(\infty) = o$. Relation (1) is uncomfortable in implementation, so after some derivations we can obtain a better formula:

$$p' = I_S(p) = o + \left[\frac{d_e(o, b)}{d_e(o, p)}\right]^2 (p - o). \tag{2}$$

Now, having a set of k star shaped sets that define star-shaped set inversion transformations we are able to generate an inversion fractal. For this purpose we can use algorithm presented in Algorithm 1. The P_v in the algorithm is an iteration process: iteration from fixed point theory or switching process [3]. In the examples presented later in Sec. 4 we will use only the standard Picard iteration, i.e., iteration process of the form:

$$p_{i+1} = I_S(p_i). \tag{3}$$

3 Pseudoinversion Fractals

In the definition of inversion transformation (circle or star-shaped set) we use the Euclidean metric. In [8] Ramírez et al. have changed the metric to the metrics:

$$d_q(a, b) = \left(|a_x - b_x|^q + |a_y - b_y|^q\right)^{\frac{1}{q}}, \tag{4}$$

where $a, b \in \mathbb{R}^2$ and $q \in [1, \infty)$. So, using this modification the inversion transformation has the following form:

$$I_{S,q}(p) = o + \left[\frac{d_q(o, b)}{d_q(o, p)}\right]^2 (p - o), \tag{5}$$

Algorithm 1: Extended random inversion algorithm with colouring [3]

Input: $S_1, \ldots, S_k$ – star-shaped sets with chosen centres of inversion, $c_1, \ldots, c_k$ – colours of the transformations, p_0 – starting point external to $S_1, \ldots, S_k$, $n > 20$ – number of iterations, P_v – iteration with parameters v, W, H – image dimensions, $\gamma \in \mathbb{R}_+$

Output: Image I with an approximation of a star-shaped set inversion fractal

for $(x, y) \in \{0, 1, \ldots, W-1\} \times \{0, 1, \ldots, H-1\}$ **do**
  $I(x, y) = \text{black}$
  $\mathcal{H}(x, y) = 0$
$c = \text{random colour}$
$j = \text{random number from } \{1, \ldots, k\}$
$p = P_v(I_{S_j}, p_0)$
for $i = 2$ **to** n **do**
  $l = \text{random number from } \{1, \ldots, k\}$
  while $j = l$ **or** $inSet(S_l, p)$ **do**
    $l = \text{random number from } \{1, \ldots, k\}$
  $j = l$
  $p = P_v(I_{S_j}, p)$
  if $i > 20$ **then**
    $x = \lfloor x_p \rfloor$
    $y = \lfloor y_p \rfloor$
    $\mathcal{H}(x, y) = \mathcal{H}(x, y) + 1$
    $c = \frac{c + c_j}{2}$
    $I(x, y) = c$
$m_{\mathcal{H}} = \max_{(x,y)} \mathcal{H}(x, y)$
for $(x, y) \in \{0, 1, \ldots, W-1\} \times \{0, 1, \ldots, H-1\}$ **do**
  if $\mathcal{H}(x, y) > 0$ **then**
    $I(x, y) = \left(\frac{\log_2(1 + \mathcal{H}(x, y))}{\log_2(1 + m_{\mathcal{H}})} \right)^{1/\gamma} I(x, y)$

where $q \in [1, \infty)$.

In the case of circle inversion together with the change of the metric the shape of the circle also changes, so the value of the inversion is different in different metric spaces. But, in the case of the star-shaped sets the shape of the set remains unchanged and it is easy to prove the following theorem.

Theorem 1. *Let S be a star-shaped set, $o \in \ker S$ be a centre of inversion and p point for which we want to calculate the inverse. Assume that b is the point of intersection of $r(t) = o + t(p - o)$, where $t \in [0, \infty)$ with the boundary of S. Then, for any $q_1, q_2 \in [1, \infty)$:*

$$\frac{d_{q_1}(o, b)}{d_{q_1}(o, p)} = \frac{d_{q_2}(o, b)}{d_{q_2}(o, p)}. \tag{6}$$

From Theorem 1 we can conclude that for a fixed star shaped set S and any $q_1, q_2 \in [1, \infty)$ the following equality is true:

$$\forall_{p \in \mathbb{R}^2} \quad I_{S,q_1}(p) = I_{S,q_2}(p). \tag{7}$$

So, the use of different metrics of the form (4) does not change the value of the star shaped inversion transformation and thus the inversion fractal remains the same.

From mathematical analysis we know that for any $q_1, q_2 \in [1, \infty)$ such that $q_1 \leq q_2$ we have [7]:

$$d_{q_2}(a, b) \leq d_{q_1}(a, b). \tag{8}$$

From this fact we can conclude that for $q_1 \neq q_2$ ($q_1, q_2 \in [1, \infty)$) and for a fixed $q \in \{q_1, q_2\}$ we have:

$$\frac{d_q(o, b)}{d_q(o, p)} \leq \frac{d_{q_1}(o, b)}{d_{q_2}(o, p)} \quad \text{or} \quad \frac{d_q(o, b)}{d_q(o, p)} \geq \frac{d_{q_1}(o, b)}{d_{q_2}(o, p)}. \tag{9}$$

In the inversion transformation we can use a pair of metrics for q_1 and q_2 ($q_1 \neq q_2$) instead of one metric for q. In this way, following (9), we change the value of the inversion transformation. The obtained point will be laying (on the ray) closer or further from the centre of inversion. This modification of inversion transformation causes that we loose some of the properties of the inversion. Because of that the modified inversion transformation will be called pseudoinversion transformation.

Replacing the inversion transformations with pseudoinversions will change the shape of the original inversion fractal. This type of fractal will be called pseudoinversion fractal.

If we look at the set of inversion transformations as the transformations in separate metric spaces, then for each of the transformations we can use a different pair of metrics (q_1, q_2). This will allow us to modify the shape of the fractal in a local manner.

Moreover, we can introduce a switching process of the metric spaces. Let us assume that we have M pairs of numbers defining metrics of the form (4), i.e., $(q_1^0, q_2^0), (q_1^1, q_2^1), \ldots, (q_1^{M-1}, q_2^{M-1})$. Now, in the m-th iteration of the iteration process we use $m \mod M$ pair of metric spaces, i.e., $(q_1^{m \mod M}, q_2^{m \mod M})$.

4 Examples

In this section, we present some examples of pseudoinversion fractals obtained with the proposed methods. The first example is presenting the use of pseudoinversion transformation using one pair $(q_1, q_2) \in [1, \infty)^2$ of parameters defining metrics for all the transformations. Fig. 1 presents the star-shaped sets defining the transformations and the inversion fractal generated using the inversion transformations of the sets. Examples of pseudoinversion fractals generated with the same star-shaped sets are presented in Fig. 2. The parameters used to generate these images were the following (from left): $(3, 7)$, $(2, 1)$, $(10, 3)$. From the

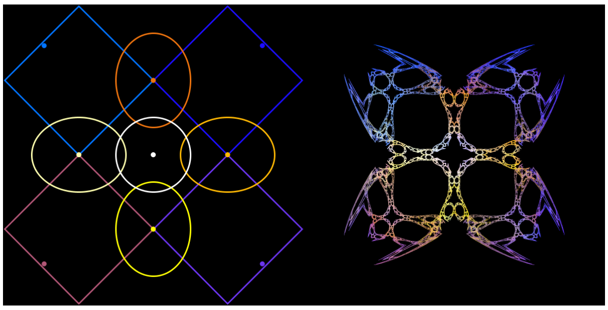

Fig. 1. Star-shaped sets defining the transformations (left) and original inversion fractal (right)

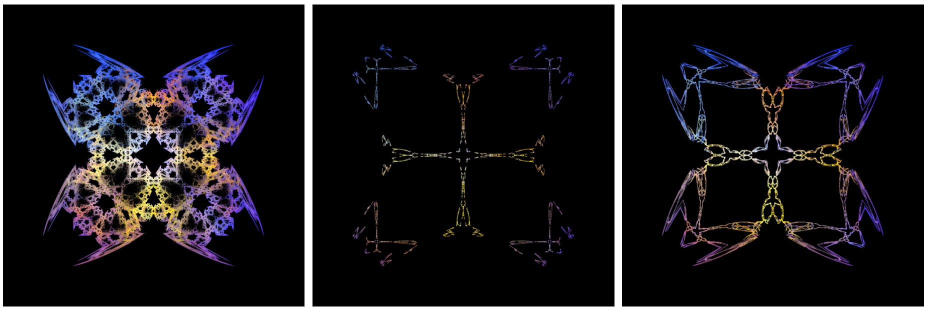

Fig. 2. Pseudoinversion fractals obtained with the use of different metrics

images we see that using different pairs of metrics we are able to obtain new fractal shapes that are different from the original inversion fractal.

In the second example we will use the same star-shaped sets and different pairs of metrics for different sets. Star-shaped sets defining the transformations and original inversion fractal are presented in Fig. 3. Fig. 4 presents examples of psuedoinversion fractals. The pairs of metrics for the individual sets are gathered in Tab. 1. From the figure we can observe that the use of different pairs of metrics for different transformations changes the shape of the fractal. In this way we can place the sets in a symmetrical way and the shape of the fractal can loose its symmetry, e.g., left image in Fig. 4. Moreover, we can observe that the shapes of pseudoinversion fractals differ in a significant way from the original inversion fractal.

The last example present fractal shapes obtained with the switching process of metrics. Fig. 5 presents star-shaped sets defining the transformations and original inversion fractal. In the first example of switching we will use two pairs of metrics. Images on the left and in the middle of Fig. 6 present pseudoinversion fractals obtained with the pairs: $(1, 3)$, $(2, 1)$, respectively. Fractal pattern obtained using switching process of these two pairs of metrics is presented on the right of Fig. 6.

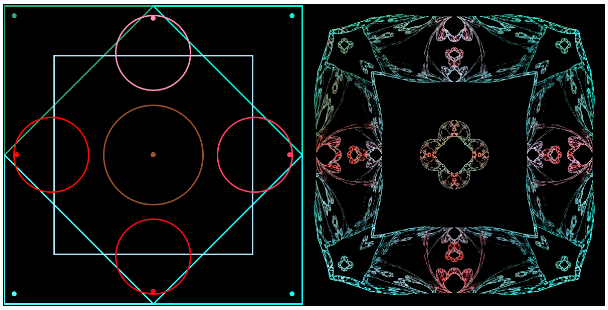

Fig. 3. Star-shaped sets defining the transformations (left) and original inversion fractal (right)

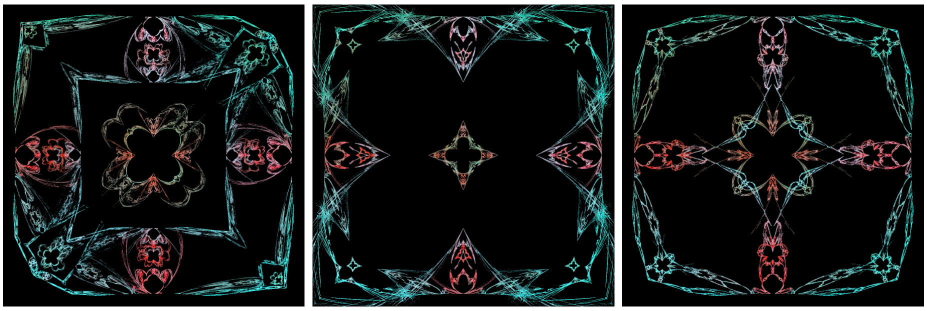

Fig. 4. Pseudoinversion fractals obtained with the use of various metrics for different transformations

Table 1. Parameters used to generate fractals from Fig. 4, T – triangle, Sq – square, C – circle, N – North, S – South, E – East, W – West, M – middle

Image	NWT	NET	SET	SWT	Sq	NC	EC	SC	WC	MC
Left	$(2,1)$	$(2,3)$	$(2,1)$	$(2,3)$	$(2,2)$	$(2,2)$	$(2,2)$	$(2,2)$	$(2,2)$	$(1.1,3)$
Middle	$(2,2)$	$(2,2)$	$(2,2)$	$(2,2)$	$(1,2.3)$	$(2,2)$	$(2,2)$	$(2,2)$	$(2,2)$	$(2,2)$
Right	$(2,2)$	$(2,2)$	$(2,2)$	$(2,2)$	$(5,1)$	$(2,2)$	$(2,2)$	$(2,2)$	$(2,2)$	$(1,5)$

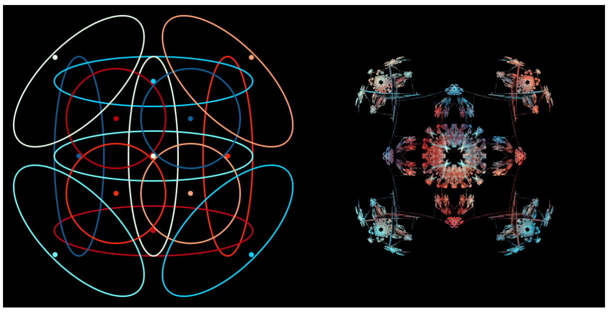

Fig. 5. Star-shaped sets defining the transformations (left) and original inversion fractal (right)

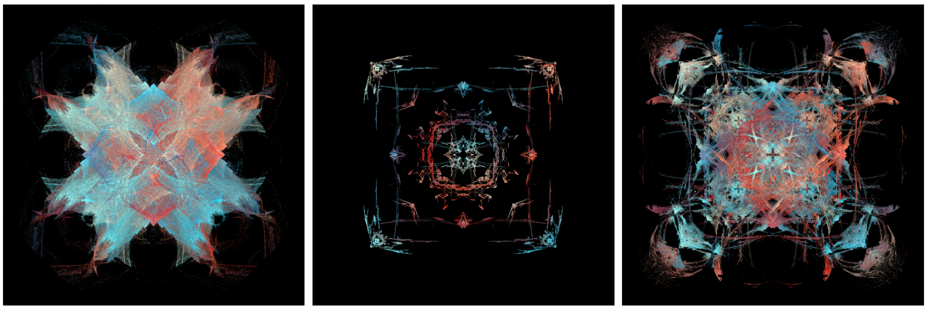

Fig. 6. Original pseudoinversion fractals (left, middle) and the result of switching their metric spaces (right)

The second example of switching metric spaces in presented in Fig. 7. This time we switch between three different pairs of metric spaces. The patterns at the top of this figure were obtained using the following pairs: $(2, 3)$, $(3, 1)$, $(5, 3)$. The result of switching between these three pairs of metric spaces is presented in the bottom part of Fig. 7.

5 Conclusions

In this paper, we presented modification of inversion fractals. The proposed modification was based on the use of different metrics in the inversion transformation formula. Moreover, we proposed a switching process between different metric spaces. Patterns which were obtained with the proposed modification differ in a significant way from the original inversion fractals and form new fractal shapes. Because of the interesting and aesthetic structure the pseudoinversion fractals can be used among other things as textile, wallpaper or ceramics patterns.

References

1. Frame, M., Cogevina, T.: An Infinite Circle Inversion Limit Set Fractal. Computers & Graphics 24(5), 797–804, (2000)
2. Gdawiec, K.: Star-shaped Set Inversion Fractals. Fractals 22(4), 1450009, 7 pages, (2014)
3. Gdawiec, K.: Inversion Fractals and Iteration Processes in the Generation of Aesthetic Patterns. Computer Graphics Forum, DOI: 10.1111/cgf.12783, (in press)
4. Lu, J., Zou, Y., Tu, G., Wu, H.: A Family of Functions for Generating Colorful Patterns with Mixed Symmetries from Dynamical Systems. In: Wang, W. (ed.) Mechatronics and Automatic Control Systems. Lecture Notes in Electrical Engineering, vol. 237, pp. 883–890. Springer, Switzerland (2014)
5. Mitchell, K.: Fun with Chaotic Orbits in the Mandelbrot Set. In: Bridges 2012, pp. 389–392. Towson, USA (2012)

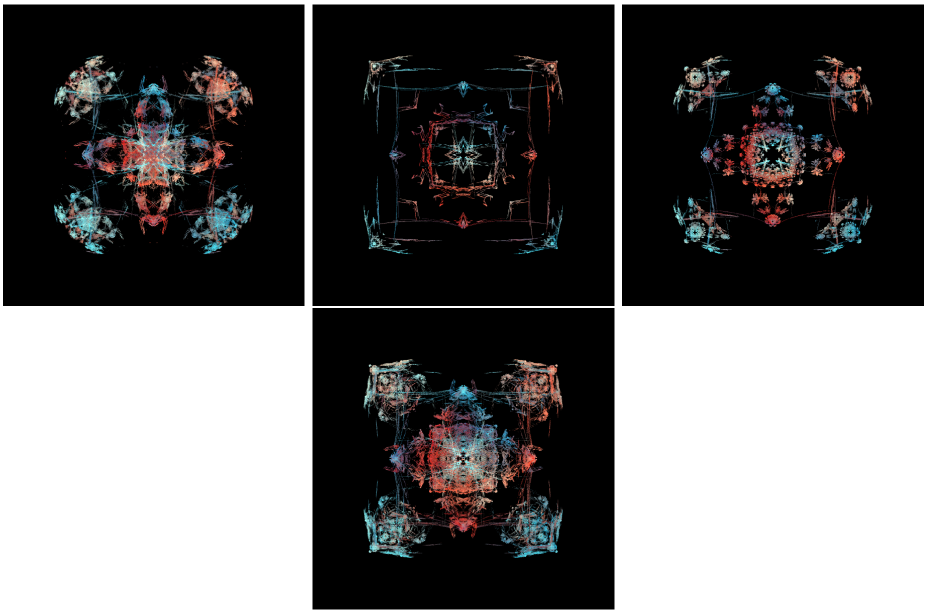

Fig. 7. Original pseudoinversion fractals (top) and the result of switching their metric spaces (bottom)

6. Ouyang, P., Cheng, D., Cao, Y., Zhan, X.: The Visualization of Hyperbolic Patterns from Invariant Mapping Method. Computers & Graphics 36(2), 92–100, (2012)
7. Raïssouli, M., Jebril, I.H.: Various Proofs for the Decrease Monotonicity of the Schatten's Power Norm, Various Families of $\mathbb{R}^n$-norms and Some Open Problems. International Journal of Open Problems in Computer Science and Mathematics 3(2), 164–174, (2010)
8. Ramírez, J.L., Rubiano, G.N., Zlobec, B.J.: Generating Fractal Patterns by Using p-Circle Inversion. Fractals 23(4), 1550047, 13 pages, (2015)
9. van Loocke, P.: Polygon-based Fractals from Compressed Iterated Function Systems. IEEE Computer Graphics and Applications 30(2), 34–44, (2010)