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ON INVERTIBLE PRESERVERS OF SINGULARITY AND NONSINGULARITY OF MATRICES OVER A FIELD

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Abstract. Invertible operators preserving singularity of matrices were studied in [3] and [4] under assumption that operators are linear. In the present paper the linearity of operators is not assumed: we assume only that operators are of the form $F = (f_{i,j})$, where $f_{i,j} : \mathcal{F} \longrightarrow \mathcal{F}$ and $\mathcal{F}$ is a field, $i, j \in \{1, 2, \ldots, n\}$. If $n \geq 3$, then in the matrix space $M_n(\mathcal{F})$ operators preserving singularity of matrices must be as in [1]. If $n \leq 2$, then operators may be nonlinear. In this case the forms of the operators are presented.

Let $\mathbb{R}, \mathbb{C}, \mathbb{N}$ denote the set of real numbers, complex numbers or positive integer numbers, respectively. Let $M_n(\mathcal{F})$ be the set of $n \times n$ matrices over a field $\mathcal{F}$, i.e. $M_n(\mathcal{F}) = \mathcal{F}^{n\times n}$, where $n \in \mathbb{N}$.

First of all let us introduce

Definition 1. An operator F from $M_n(\mathcal{F})$ into itself is an operator preserving singularity of matrices from $M_n(\mathcal{F})$ if and only if for every singular matrix $A \in M_n(\mathcal{F})$ the matrix $F(A)$ is singular.

Definition 2. An operator F from $M_n(\mathcal{F})$ into itself is an operator preserving nonsingularity of matrices from $M_n(\mathcal{F})$ if and only if for every nonsingular matrix $A \in M_n(\mathcal{F})$ the matrix $F(A)$ is nonsingular.

Let S, NS denote the set of singular or nonsingular matrices from $M_n(\mathcal{F})$, respectively.

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In the paper we consider the operators F from $M_n(\mathcal{F})$ into itself of the form

$$F = (f_{i,j}), \quad \text{where} \quad f_{i,j}\colon \mathcal{F} \longrightarrow \mathcal{F}, \quad i,j = 1,2,\dots,n, \tag{1}$$

where the matrix $F(A) := (f_{i,j}(a_{i,j}))$ for $i,j = 1,2,\dots,n$, for any matrix $A \in M_n(\mathcal{F})$.

Remark 1. In case of $n = 1$ an invertible operator F of the form (1) is an operator preserving singularity of matrices from $M_1(\mathcal{F})$ if and only if for $x \in \mathcal{F}$ the equivalence $x = 0 \iff f_{1,1}(x) = 0$ holds.

Lemma 1. *If an operator F of the form* (1)*, from $M_n(\mathcal{F})$ into itself for $n \geq 2$, where $\mathcal{F}$ is a field, is an invertible operator, then all functions $f_{i,j}$ for $i,j \in \{1,2,\dots,n\}$ are injective.*

Proof. Let us assume that $n \geq 2$. We denote the matrix whose i,j entry is 1 and the remaining entries of which are 0 by $E_{i,j}$. Let us consider the matrices $F(xE_{i,j})$. If F is an invertible operator, then the function $f_{i,j}$ is injective, which completes the proof. □

Lemma 2. *If an invertible operator F of the form* (1)*, from $M_n(\mathcal{F})$ into itself for $n \geq 2$, where $\mathcal{F}$ is a field, preserves singularity of matrices in the space $M_n(\mathcal{F})$, then the equivalence*

$$x = 0 \quad \iff \quad f_{i,j}(x) = 0 \tag{2}$$

holds for all $x \in \mathcal{F}$, $i,j \in \{1,2,\dots,n\}$.

Proof. Let us assume that $n \geq 2$. Let F be an operator preserving singularity of matrices. Let indices $i,j \in \{1,2,\dots,n\}$ be arbitrary and fixed. We would like to prove that $f_{i,j}(0) = 0$.

Let us consider the matrix $B_1 = (b_{k,l})$, $k,l = 1,2,\dots,n$, such that $b_{i,l} = 0$ for $l = 1,2,\dots,n$. When we exchange the first row with the i-th row and next exchange first column with the j-th column, we obtain the matrix $B_2 \in S$. Then also $F(B_2) \in S$ and with $f_{i,j}(0)$ in the left-upper corner.

Therefore without the loss of generality we may prove that $f_{1,1}(0) = 0$. Let us consider the matrix B_3 with all elements in the first row being equal to zero. Then $F(B_3) \in S$. Denote $y_{k,l} := f_{k,l}(b_{k,l})$ for $k,l = 1,2,\dots,n$.

We contradictory assume that $f_{1,1}(0) \neq 0$.

We build singular matrices B_3^k and Y^k obtained from the singular matrix $F(B_3^k)$, for $k = 1,2,\dots,n$, by operations do not changing the determinant of $F(B_3^k)$.

As a first step we define the matrix $B_3^1 := B_3$. We construct Y^1 with entries $y_{k,l}^1$. We obtain the matrix $F(B_3)$ with $y_{1,1} = f_{1,1}(0) \neq 0$. Multiplying the first column of this matrix by $y_{1,1}^{-1} \cdot y_{1,2}$, $y_{1,1}^{-1} \cdot y_{1,3}$, $\dots$, $y_{1,1}^{-1} \cdot y_{1,n}$ and subsrtacting from the second, third, $\dots$, n-th column, respectively, we obtain the first row with entries equals to $y_{1,1}, 0, \dots, 0$. Next, multiplying the first row by $y_{k,1}^1 \cdot y_{1,1}^{-1}$ and substracting from the second, third, $\dots$, n-th row we obtain that $y_{k,1} = 0$ for $k = 2, 3 \dots, n$. Then in the k-th row, $k = 2, 3, \dots, n$ the obtained entries are:

$$0,\ y_{k,2} - y_{1,2}y_{1,1}^{-1}y_{k,1},\ y_{k,3} - y_{1,3}y_{1,1}^{-1}y_{k,1}, \dots,\ y_{k,n} - y_{1,n}(y_{1,1}^1)^{-1}y_{k,1}.$$

The obtained matrix we denote by Y^1 and its entries by $y_{k,l}^1$.

As a second step let us consider the element $y_{2,2}^1 = y_{2,2} - y_{1,1}^{-1}y_{2,1}$. If $y_{2,2}^1 \neq 0$ then $B_3^2 := B_2^1$. If $y_{2,2}^1 = 0$ then B_3^2 is the matrix obtained from B_3^1 with replaced element $b_{2,2}^1$ by $\overline{b_{2,2}^1} \in \mathcal{F}$, $\overline{b_{2,2}^1} \neq b_{2,2}^1$. Let us define $y_{2,2}^2 = f_{2,2}(\overline{b_{2,2}^1})$. As $f_{2,2}$ is an injective function, then $y_{2,2}^2 \neq 0$. Using this element we bring to zero the elements of the second row and next the second column. We denote the obtained matrix in this way by Y^2; it is a singular matrix. We can see that $y_{k,l}^2 = y_{k,l}^1 - y_{k,2}^1(y_{2,2}^2)^{-1}y_{2,l}^1$ for $k, l = 3, 4, \dots, n$.

In the r-th step for $r = 3, 4, \dots, n-1$ we consider the element $y_{r,r}^{r-1} = y_{2,2}^{r-2} - (y_{1,1}^{r-1})^{-1}y_{2,1}^{r-1}$. If $b_{r,r}^{r-1} = 0$ then $B_3^r := B_3^{r-1}$. If $y_{r,r}^{r-1} = 0$, then B_3^r is the matrix obtained from B_3^{r-1} with replace the element $b_{r,r}^{r-1}$ by $\overline{b_{r,r}^{r-1}} \in \mathcal{F}$, $\overline{b_{r,r}^{r-1}} \neq b_{r,r}^{r-1}$. We define $y_{r,r}^r = f_{r,r}(\overline{b_{r,r}^{r-1}})$. As $f_{r,r}$ is an injective function, then $y_{r,r}^r \neq 0$. Using this element we bring to zero the elements of the second row and next the second column. The obtained matrix in this way we denote by Y^r; it is a singular matrix. We can see that $y_{k,l}^r = y_{k,l}^{r-1} - y_{k,2}^1(y_{2,2}^2)^{-1}y_{2,l}^{r-1}$ for $k, l = r+1, r+2, \dots, n$.

In the last n-th step we consider the element $y_{n,n}^{n-1} = y_{n,n}^{n-2} - (y_{1,1}^{n-1})^{-1}y_{2,1}^{n-1}$. If $y_{n,n}^{n-1} \neq 0$ then $B_3^n := B_3^{n-1}$ and $Y^n := Y^{n-1}$. If $y_{n,n}^{n-1} = 0$ then we replace $b_{n,n}^{n-1}$ by $\overline{b_{n,n}^{n-1}} \in \mathcal{F}$, $\overline{b_{n,n}^{n-1}} \neq b_{n,n}^{n-1}$. Then the matrix Y^n is obtained from the matrix Y^{n-1} replacing the element $y_{n,n}^n$ by $\overline{y_{n,n}^n} = f_{n,n}(\overline{b_{n,n}^{n-1}})$.

The Y^n is a diagonal matrix $Y^n = \operatorname{diag}(f_{1,1}(0), y_{2,2}^2, y_{3,3}^3, \dots, y_{n,n}^n)$, where $y_{r,r}^r \neq 0$ for $r = 2, 3, \dots, n$.

Now, taking instead of the matrix B_3 the singular matrix B_3^n and carrying out similar operations on matrices, we obtain the same singular diagonal matrix Y^n with the determinant det $(Y_n) = f_{1,1}(0) \cdot y_{2,2}^2 \cdot y_{3,3}^3 \cdot \dots \cdot y_{n,n}^n = 0$. As $y_{r,r}^r \neq 0$ for $r = 2, 3, \dots, n$, then $f_{1,1}(0) = 0$. It is contradictory with the assumption.

By Lemma 1 $f_{i,j}$ is an injective function and therefore $f_{i,j}(x) \neq 0$ for $x \neq 0$, which completes the proof. □

An important role in determining preservers of matrices is played by the functions satisfying simultaneously the multiplicative and additive Cauchy functional equations. In particular cases ($\mathcal{F} = \mathbb{C}$ or $\mathcal{F} = \mathbb{R}$) the following holds true.

Remark 2 (see [6], Chapter XIV, §4, 5 and 6). In the case $\mathcal{F} = \mathbb{C}$ there are infinitely many functions g fulfilling simultaneuosly the multiplicative Cauchy functional equation $g(xy) = g(x)g(y)$ also the additive Cauchy functional equation $g(x+y) = g(x) + g(y)$. In the case $\mathcal{F} = \mathbb{R}$ there are two solutions: $g = \mathrm{id}$ and $g \equiv 0$.

We prove the main result of the paper

Theorem. (a) *If $n = 2$, then an invertible operator F preserves the singularity of matrices on $M_n(\mathcal{F})$ if and only if there exist nonzero $u_1, u_2, v_1, v_2 \in \mathcal{F}$ and an injective function $g\colon \mathcal{F} \longrightarrow \mathcal{F}$ satisfying $g(0) = 0$ and $g(xy) = g(x)g(y)$ for all $x, y \in \mathcal{F}$ such that $f_{i,j}(x) = u_i v_j g(x)$ for all $x \in \mathcal{F}$.*

(b) *If $n \geq 3$, then an invertible operator F preserves the singularity of matrices on $M_n(\mathcal{F})$ if and only if there are nonzero $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \in \mathcal{F}$ and an injective function $g\colon \mathcal{F} \longrightarrow \mathcal{F}$ satisfying $g(xy) = g(x)g(y)$ and $g(x+y) = g(x) + g(y)$ for all $x, y \in \mathcal{F}$ such that $f_{i,j}(x) = u_i v_j g(x)$ for all $x \in \mathcal{F}$.*

Thus, for $n \geq 3$ the singularity preserving maps F on $M_n(\mathcal{F})$ may be written in the form of $F(A) = U[g(a_{i,j})]V$, where $U = \mathrm{diag}(u_1, u_2, \ldots, u_m)$ and $V = \mathrm{diag}(v_1, v_2, \ldots, v_n)$ are invertible diagonal matrices and g is an injective endomorphism of $\mathcal{F}$. For $n = 2$, the additivity of g may even be relaxed to the sole requirement that $g(0) = 0$. Note that the maps of part (a) may be nonlinear: for example, one can take $g(x) = x^3$.

Proof. Let $n \geq 2$ and suppose F is an operator preserving singularity on $M_n(\mathcal{F})$. By Lemma 2 $f_{i,j}(0) = 0$ and $f_{i,j}(x) \neq 0$ for all i, j and all $x \neq 0$. Denote $c_{i,j} = f_{i,j}(1)$. Because $f_{i,j}(1) \neq 0$, we obtain that $c_{i,j} \neq 0$ for all i, j. Put $g_{i,j}(x) = c_{i,j}^{-1} f_{i,j}(x)$. Clearly, $g_{i,j}(0) = 0$ and $g_{i,j}(1) = 1$ for all i, j. Let us define the matrix $C = (c_{i,j})$. By Lemma 2, the rank $C \geq 1$. We prove that rank $C = 1$.

In the case $n = 2$ the matrix whose all entries are 1 is singular, then rank $C = 1$.

In the case $n \geq 3$ we suppose contradictory that rank $C = k > 1$. Without any loss of generality we may assume that the determinant of the upper-left submatrix of C of the order k is not equal to zero. Let us consider the matrix $B_1 = \sum_{i=1}^{k} \sum_{j=1}^{k} E_{i,j} + \sum_{l=k+1}^{n} E_{l,l}$. Note that the matrix $B_1 \in S$. Then the matrix $F(B_1) = \sum_{i=1}^{k}\sum_{j=1}^{k} E_{i,j}c_{i,j} + \sum_{l=k+1}^{n} c_{l,l}E_{l,l}$. From the properties of determinants det $F(B_1) \neq 0$ and $F(B_1) \in S$. Then the rank C can not be greater than 1, it must be equal to one.

Then in cases $n = 2$ and $n \geq 3$ the equality rank $C = 1$ holds. This implies that there are $u_i, v_j \in \mathcal{F}$ such that $f_{i,j}(1) = u_i v_j$ for all i, j.

For $1 \leq i \neq r \leq n$ and $1 \leq k \neq l \leq n$, let $B_2 = E_{i,k} + xE_{i,l} + E_{r,k} + xE_{r,l}$. Matrices $B_2, F(B_2) \in S$ and therefore

$$0 = f_{i,k}(1)f_{r,l}(x) - f_{r,k}(1)f_{i,l}(x) = u_i v_k u_r v_l g_{r,l}(x) - u_r v_k u_i v_l g_{i,l}(x),$$

as $g_{r,l}(x) = g_{i,l}(x)$ for all $x \in \mathcal{F}$. Consequently, the matrix $G = (g_{i,j})$ is constant along its column. Analogously one can show that G is constant along the rows. This implies that all $g_{i,j}$ are one and the same function g and that therefore $f_{i,j}(x) = u_i v_j g(x)$ for all i, j and all x. Note that $g(0) = 0$ and $g(1) = 1$.

By Lemma 1 we obtain that g is an injective function.

To show that $g(xy) = g(x)g(y)$, take $B_3 = E_{1,1} + xE_{1,2} + yE_{2,1} + xyE_{2,1}$. Since $B_3, F(B_3) \in S$, we obtain that

$$0 = f_{1,1}(1)f_{2,2}(xy) - f_{1,2}(x)f_{2,1}(y) = u_1 v_1 u_2 v_2 g(xy) - u_1 v_2 u_2 v_1 g(x)g(y),$$

that is, we arrive at the equality $g(xy) = g(x)g(y)$ for all $x, y \in \mathcal{F}$.

At this point we have proved the necessary condition on F part of (a). To obtain the necessary condition on F part of (b), we assume $n \geq 3$ and consider

$$B_4 = xE_{1,1} + E_{1,2} + yE_{2,1} + E_{2,3} + (x + y)E_{3,1} + E_{3,2} + E_{3,3}.$$

As $B_4 \in S$, we conclude that the determinant of the upper-left 3×3 submatrix of $F(B_4)$ must be zero, which means that

$$0 = u_1 u_2 u_3 v_1 v_2 v_3(-g(x) - g(y) + g(xy)).$$

Thus, $g(x + y) = g(x) + g(y)$. The proof of the necessary condition on F part of (b) is also complete.

We now prove the sufficient condition on the F parts (a) and (b). By Lemma 2 F maps the zero matrix to itself. By Theorem from [5] it follows that F is an operator preserving rank of matrices from $M_n(\mathcal{F})$ in parts (a) and (b). Then it also preserves the singularity of matrices from $M_n(\mathcal{F})$, which completes the proof. □

An analogous theorem for invertible operators of the form (1) preserving nonsingularity of matrices is not true. Let us consider the following example.

Example. In particular case $\mathcal{F} = \mathbb{R}$ let us consider the operator $H = (h_{i,j})$ of the form (1) from $M_n(\mathbb{R})$ into itself with functions

$$h_{i,j}(x) = \begin{cases} n!\left(\frac{7}{4} + \frac{1}{2\pi}\arctan(x)\right) & \text{for } i = j, \\ \frac{1}{n!}\left(\frac{3}{4} + \frac{1}{2\pi}\arctan(x)\right) & \text{for } i \neq j \end{cases}$$

for $x \in \mathbb{R}$. The functions $h_{i,j}$ are injective on $\mathbb{R}$.

Let us consider a matrix $X \in M_n(\mathbb{R})$ with entries $x_{i,j}$. We prove that H maps every matrix from $M_n(\mathbb{R})$ to NS. We prove that the determinant of the matrix $H(X)$ is positive.

From the definition of the determinant

$$\det H(X) = \prod_{i=1}^{n} h_{i,i}(x_{i,i}) + \sum_{i=1}^{n} \prod_{\sigma(i)} (-1)^{I_i} h_{i,\sigma(i)}(x_{i,\sigma(i)}),$$

where σ is a permutation of the set $\{1, 2, \ldots, n\}$, I_i denotes the number of inverses in the permutation $\sigma(i)$.

Let us observe that $\frac{1}{2n!} < h_{i,j}(x_{i,j}) < \frac{1}{n!}$ for $i \neq j$ and $n! < h_{i,i}(x_{i,i}) < 2n!$. From the above inequalities $\prod_{i=1}^{n} h_{i,i}(x_{i,i}) > (n!)^n$ and

$$\prod_{\sigma(i)} h_{i,\sigma(i)}(x_{i,\sigma(i)}) < \frac{1}{n!} \cdot (2n!)^{n-1} = 2^{n-1} \cdot (n!)^{n-2}.$$

Using these inequalities we obtain

$$\begin{aligned} \det H(X) &> (n!)^n + \left(\frac{n!}{2} - 1\right) \cdot \left(\frac{1}{2n!}\right)^n - \left(\frac{n!}{2}\right) \cdot (n!)^{n-2} \cdot 2^{n-1} \\ &> (n!)^n - (n!)^{n-1} \cdot 2^{n-2} = (n!)^{n-1}(n! - 2^{n-2}). \end{aligned}$$

As $n! - 2^{n-2} > 0$ for $n \in \mathbb{N}$, then $\det H(X) > 0$, i.e. $H(X) \in NS$.

The operator H is invertible and preserves the nonsingularity of matrices from $M_n(\mathbb{R})$.

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References

[1] Li C.K., Pierce S., *Linear preserver problems*, Amer. Math. Monthly **108** (2001), 591–605.

[2] Dieudonné J., *Sur une généralisation du groupe orthogonal à quatre variables*, Arch. Math. **1** (1949), 282–287.

[3] Guralnick R.M., *Invertible preservers and algebraic groups. II. Preservers of similarity invariants and overgroups of* $PSL_n(\mathbf{F})$, Linear Multilinear Algebra **43** (1997), 221–255.

[4] Guralnick R.M., Li C.K., *Invertible preservers and algebraic groups. III. Preservers of unitary similarity (congruence) invariants and overgroups of some unitary groups*, Linear Multilinear Algebra **43** (1997), 257–282.

[5] Kalinowski J., *Preservers of the rank of matrices over a field*, Beiträge Algebra Geom. **50** (2009), 215–218.

[6] Kuczma M., *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*, Uniwersytet Śląski, Katowice, Polish Sci. Publ., Warsaw, 1985.

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