Title: On a problem of Janusz Matkowski and Jacek Wesołowski

Author: Janusz Morawiec, Thomas Zürcher

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Abstract. We study the problem of the existence of increasing and continuous solutions \( \varphi : [0, 1] \rightarrow [0, 1] \) such that \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \) of the functional equation

\[
\varphi(x) = \sum_{n=0}^{N} \varphi(f_n(x)) - \sum_{n=1}^{N} \varphi(f_n(0)),
\]

where \( N \in \mathbb{N} \) and \( f_0, \ldots, f_N : [0, 1] \rightarrow [0, 1] \) are strictly increasing contractions satisfying the following condition \( 0 = f_0(0) < f_0(1) = f_1(0) < \cdots < f_{N-1}(1) = f_N(0) < f_N(1) = 1 \).

In particular, we give an answer to the problem posed in Matkowski (Aequationes Math. 29:210–213, 1985) by Janusz Matkowski concerning a very special case of that equation.

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1. Introduction

During the 47th International Symposium on Functional Equations in 2009 Jacek Wesołowski asked whether the identity on \([0, 1]\) is the only increasing and continuous solution \( \varphi : [0, 1] \rightarrow [0, 1] \) of the equation

\[
\varphi(x) = \varphi\left(\frac{x}{2}\right) + \varphi\left(\frac{x+1}{2}\right) - \varphi\left(\frac{1}{2}\right)
\]

(satisfying

\[
\varphi(0) = 0 \quad \text{and} \quad \varphi(1) = 1.
\]

This question was posed in connection with studying probability measures in the plane which are invariant by “winding” (see [10]).

A negative answer to this question was obtained in [5] and reads as follows.

Theorem 1.1. (i) The identity on \([0, 1]\) is the only increasing and absolutely continuous solution \( \varphi : [0, 1] \rightarrow [0, 1] \) of Eq. \((e_1)\) satisfying \((1)\).
(ii) For every $p \in (0, 1)$ the function $\varphi_p : [0, 1] \to [0, 1]$ given by

$$\varphi_p \left( \sum_{k=1}^{\infty} \frac{x_k}{2^k} \right) = \sum_{k=1}^{\infty} x_k p^{k-\sum_{i=1}^{k-1} x_i} (1-p)^{\sum_{i=1}^{k-1} x_i},$$

where $x_k \in \{0, 1\}$ for all $k \in \mathbb{N}$, is an increasing and continuous solution of Eq. (e1) satisfying (1). Moreover, $\varphi_p$ is singular for every $p \neq \frac{1}{2}$.

Let us note that the first assertion of Theorem 1.1 is known (see e.g. [13] or [8]), however in [5] we can find an independent proof of it.

It turns out that in 1985 Janusz Matkowski posed a problem asking if Eq. (e1) has a non-linear monotonic and continuous solution $\varphi : [0, 1] \to \mathbb{R}$ (see [9]). Moreover, he observed that monotonic solutions of Eq. (e1) are connected with invariant measures for a certain map on $[0, 1]$. Note that Matkowski’s problem is equivalent to Wesolowski’s question.

Remark 1.2. (i) If $\varphi : [0, 1] \to [0, 1]$ is an increasing and continuous solution of Eq. (e1) satisfying (1), then for all $a, b \in \mathbb{R}$ the function $a \varphi + b$ is monotonic, continuous and satisfies (e1) for every $x \in [0, 1]$.

(ii) If $\varphi : [0, 1] \to \mathbb{R}$ is a monotonic and continuous solution of Eq. (e1), different from a constant function, then $\frac{\varphi - \varphi(0)}{\varphi(1) - \varphi(0)}$ is an increasing and continuous function satisfying (1) and (e1) for every $x \in [0, 1]$.

2. Preliminaries

Fix $N \in \mathbb{N}$, strictly increasing contractions $f_0, \ldots, f_N : [0, 1] \to [0, 1]$ such that

$$0 = f_0(0) < f_0(1) = f_1(0) < \cdots < f_{N-1}(0) = f_N(0) < f_N(1) = 1$$

and consider the functional equation

$$\varphi(x) = \sum_{n=0}^{N} \varphi(f_n(x)) - \sum_{n=1}^{N} \varphi(f_n(0))$$

for every $x \in [0, 1]$. Denote by $\mathcal{C}$ the class of all continuous and increasing solutions $\varphi : [0, 1] \to [0, 1]$ of Eq. (E) satisfying (1). Following the idea from [5] we show that $\mathcal{C}$ contains many functions, however, we manage to identify a quite large class of contractions that includes the similitudes such that there is exactly one absolutely continuous solution.

We begin with two observations showing that in many situations the class $\mathcal{C}$ is determined by two of its subclasses $\mathcal{C}_a$ and $\mathcal{C}_s$, consisting of all absolutely continuous and all singular functions, respectively.

Remark 2.1. If $\varphi_1, \varphi_2 \in \mathcal{C}$ and if $\alpha \in (0, 1)$, then $\alpha \varphi_1 + (1-\alpha)\varphi_2 \in \mathcal{C}$. 

To formulate the next remark we recall that a Lebesgue measurable function \( f: [0, 1] \rightarrow [0, 1] \) is said to be nonsingular if the set \( f^{-1}(A) \) has Lebesgue measure zero for every set \( A \subset [0, 1] \) of Lebesgue measure zero (see [6]). Observe that an invertible Lebesgue measurable function \( f \) is nonsingular if and only if its inverse \( f^{-1} \) satisfies Luzin’s condition (N).

**Remark 2.2.** Assume that all the contractions \( f_0, \ldots, f_N \) are nonsingular. Then, both the absolutely continuous and the singular parts of every element from \( C \) satisfy (E) for every \( x \in [0, 1] \).

**Proof.** Fix \( \varphi \in C \) and denote by \( \varphi_a \) and \( \varphi_s \) its absolutely continuous and singular parts, respectively. By (E), for every \( x \in [0, 1] \) we have

\[
\varphi_a(x) - \sum_{n=0}^{N} \varphi_a(f_n(x)) = -\varphi_s(x) + \sum_{n=0}^{N} \varphi_s(f_n(x)) - \sum_{n=1}^{N} \varphi(f_n(0)),
\]

and hence there exists a real constant \( c \) such that

\[
\varphi_a(x) - \sum_{n=0}^{N} \varphi_a(f_n(x)) = c \quad \text{and} \quad -\varphi_s(x) + \sum_{n=0}^{N} \varphi_s(f_n(x)) - \sum_{n=1}^{N} \varphi(f_n(0)) = c.
\]

This jointly with the fact that \( f_0(0) = 0 \) stipulated in (3) gives

\[
c = \varphi_a(0) - \sum_{n=0}^{N} \varphi_a(f_n(0)) = -\sum_{n=1}^{N} \varphi_a(f_n(0)),
\]

and in consequence

\[
\varphi_a(x) = \sum_{n=0}^{N} \varphi_a(f_n(x)) - \sum_{n=1}^{N} \varphi_a(f_n(0))
\]

and

\[
\varphi_s(x) = \sum_{n=0}^{N} \varphi_s(f_n(x)) - \sum_{n=1}^{N} \varphi_s(f_n(0))
\]

for every \( x \in [0, 1] \). \( \square \)

For all \( k \in \mathbb{N} \) and \( n_1, \ldots, n_k \in \{0, \ldots, N\} \) denote by \( f_{n_1, \ldots, n_k} \) the composition \( f_{n_1} \circ \cdots \circ f_{n_k} \). We extend the notation to the case \( k = 0 \) by letting \( f_{n_1, \ldots, n_0} \) be the identity.

**Lemma 2.3.** Let \( (n_k)_{k \in \mathbb{N}} \) be a sequence of elements of \( \{0, \ldots, N\} \). Then the sequence \( (f_{n_1, \ldots, n_k}(0))_{k \in \mathbb{N}} \) is increasing and the sequence \( (f_{n_1, \ldots, n_k}(1))_{k \in \mathbb{N}} \) is decreasing. Moreover,

\[
\lim_{k \to \infty} f_{n_1, \ldots, n_k}(y) = \lim_{k \to \infty} f_{n_1, \ldots, n_k}(z)
\]

1 The parts are unique up to a constant. For definiteness, we choose them such that \( \varphi_a(0) = \varphi_s(0) = 0 \).
for all $y, z \in [0, 1]$.

Proof. Fix a sequence $(n_k)_{k \in \mathbb{N}}$ of elements of $\{0, \ldots, N\}$ and an integer number $k \geq 2$. From (3) we have

$$0 \leq f_{n_k}(0) < f_{n_k}(1) \leq 1$$

and by the strict monotonicity of $f_{n_1, \ldots, n_{k-1}}$ we conclude that

$$f_{n_1, \ldots, n_{k-1}}(0) \leq f_{n_1, \ldots, n_k}(0) < f_{n_1, \ldots, n_k}(1) \leq f_{n_1, \ldots, n_{k-1}}(1).$$

To complete the proof it is enough to observe that for all $y, z \in [0, 1]$ and $k \in \mathbb{N}$ we have

$$|f_{n_1, \ldots, n_k}(y) - f_{n_1, \ldots, n_k}(z)| \leq f_{n_1, \ldots, n_k}(1) - f_{n_1, \ldots, n_k}(0) \leq c^k,$$

where $c \in (0, 1)$ is the largest Lipschitz constant of the given contractions $f_0, \ldots, f_N$. □

Lemma 2.4. For every $x \in [0, 1]$ there exists a sequence $(x_k)_{k \in \mathbb{N}}$ of elements of $\{0, \ldots, N\}$ such that

$$x = \lim_{k \to \infty} f_{x_1, \ldots, x_k}(0).$$

Proof. Fix $x \in [0, 1]$ and observe that according to Lemma 2.3 it is enough to show that there exists a sequence $(x_k)_{k \in \mathbb{N}}$ of elements of $\{0, \ldots, N\}$ such that

$$f_{x_1, \ldots, x_k}(0) \leq x \leq f_{x_1, \ldots, x_k}(1)$$

for every $k \in \mathbb{N}$.

By (3) there exists $x_1 \in \{0, \ldots, N\}$ such that

$$f_{x_1}(0) \leq x \leq f_{x_1}(1).$$

Thus, (5) holds for $k = 1$.

Fix $k \in \mathbb{N}$ and assume inductively that there exist $x_1, \ldots, x_k \in \{0, \ldots, N\}$ such that (5) holds. Then

$$0 \leq f_{x_1, \ldots, x_k}^{-1}(x) \leq 1$$

and by (3) there exists $x_{k+1} \in \{0, \ldots, N\}$ such that

$$f_{x_{k+1}}(0) \leq f_{x_1, \ldots, x_k}^{-1}(x) \leq f_{x_{k+1}}(1).$$

Hence

$$f_{x_1, \ldots, x_{k+1}}(0) \leq x \leq f_{x_1, \ldots, x_{k+1}}(1),$$

and the proof is complete. □
3. General case

Fix positive real numbers $p_0,\ldots,p_N$ such that

$$\sum_{n=0}^{N} p_n = 1. \quad (6)$$

Then there exists a unique Borel probability measure $\mu$ such that

$$\mu(A) = \sum_{n=0}^{N} p_n \mu(f^{-1}_n(A)) \quad (7)$$

for every Borel set $A \subset [0,1]$ (see [4]; cf. [3]). From now on the letter $\mu$ will be reserved for the unique Borel probability measure satisfying (7) for every Borel set $A \subset [0,1].$

**Lemma 3.1.** The measure $\mu$ is continuous.

**Proof.** As a first step we want to show that

$$\mu(\{f_n(0)\}) = \mu(\{f_n(1)\}) = 0 \quad (8)$$

for every $n \in \{0,\ldots,N\}.$

Applying (7) and using (3), we obtain

$$\mu(\{0\}) = \mu(\{f_0(0)\}) = \sum_{n=0}^{N} p_n \mu(\{f^{-1}_n(f_0(0))\}) = p_0 \mu(\{0\}) + \sum_{n=1}^{N} p_n \mu(\emptyset).$$

By the fact that $p_0 \in (0,1)$ we conclude that

$$\mu(\{f_0(0)\}) = \mu(\{0\}) = 0.$$

Similarly, applying (7), (3) and the fact that $p_N \in (0,1)$ we conclude that

$$\mu(\{f_N(1)\}) = \mu(\{1\}) = 0.$$

If $n \in \{1,\ldots,N\},$ then applying again (7) and (3), we obtain

$$\mu(\{f_{n-1}(1)\}) = \mu(\{f_n(0)\}) = p_{n-1} \mu(\{1\}) + p_n \mu(\{0\}) = 0.$$

Our second step is to prove that

$$\mu([f_{n_1},\ldots,n_k(0), f_{n_1},\ldots,n_k(1)]) = \prod_{n=0}^{N} p_n^{\#\{i \in \{1,\ldots,k\}: n_i = n\}} \quad (9)$$

for all $k \in \mathbb{N} \cup \{0\}$ and $n_1,\ldots,n_k \in \{0,\ldots,N\}.$

Since $\mu([0,1]) = 1,$ it follows that (9) is satisfied for $k = 0.$

Fix $k \in \mathbb{N} \cup \{0\}$ and assume that (9) holds for all $n_1,\ldots,n_k \in \{0,\ldots,N\}.$

Fix also $n_{k+1} \in \{0,\ldots,N\}.$

Note first that from (8), (3), and (7), we get

$$\mu(B) = p_n \mu(f^{-1}_n(B)) \quad (10)$$
for all \( n \in \{0, \ldots, N\} \) and Borel sets \( B \subset [f_n(0), f_n(1)] \). This jointly with (9) implies
\[
\mu([f_{n_1}, \ldots, n_{k+1}(0), f_{n_1}, \ldots, n_{k+1}(1)]) = p_{n_1} \prod_{n=0}^{N} p_n^{\# \{i \in \{2, \ldots, k+1\}: n_i = n\}} = \prod_{n=0}^{N} p_n^{\# \{i \in \{1, \ldots, k+1\}: n_i = n\}}.
\]

To prove that \( \mu \) is continuous it is sufficient to show that \( \mu \) has no atoms.

Fix \( x \in [0, 1] \). From Lemma 2.4 we conclude that there exists a sequence \((x_k)_{k \in \mathbb{N}}\) of elements of \( \{0, \ldots, N\} \) such that (4) holds. Then applying Lemma 2.3 and (9) with \( n_i = x_i \) for \( i \in \{1, \ldots, k\} \), we obtain
\[
\mu(\{x\}) = \mu\left(\bigcap_{k \in \mathbb{N}} [f_{x_1}, \ldots, x_k(0), f_{x_1}, \ldots, x_k(1)]\right)
= \lim_{k \to \infty} \mu\left([f_{x_1}, \ldots, x_k(0), f_{x_1}, \ldots, x_k(1)]\right)
= \lim_{k \to \infty} \prod_{n=0}^{N} p_n^{\# \{i \in \{1, \ldots, k\}: x_i = n\}} \leq \lim_{k \to \infty} (\max\{p_0, \ldots, p_N\})^k = 0,
\]
and the proof is complete. \( \square \)

The next lemma is folklore (the reader can consult [2,12] in the case where \( f_0, \ldots, f_N \) are similitudes and [7] in the case where \( f_0, \ldots, f_N \) are contractions). More general results in this direction can be found e.g. in [14,15].

**Lemma 3.2.** The measure \( \mu \) is either singular or absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \).

Define the function \( \varphi: [0, 1] \to [0, 1] \) by
\[
\varphi(x) = \mu([0, x]).
\]

From now on the letter \( \varphi \) will be reserved for the just defined function.

**Theorem 3.3.** Either \( \varphi \in C_a \) or \( \varphi \in C_s \).

**Proof.** We first prove that \( \varphi \in C \).

That \( \varphi \) is increasing is a consequence of the monotonicity of \( \mu \). The continuity of \( \varphi \) and that \( \varphi(0) = 0 \) follows from Lemma 3.1. Since \( \mu \) is a probability measure, we have \( \varphi(1) = 1 \).

From (10) we get
\[
\mu(f_n(B)) = p_n \mu(B)
\]
for all \( n \in \{0, \ldots, N\} \) and Borel sets \( B \subset [0, 1] \). This jointly with (6) gives
\[
\sum_{n=0}^{N} \mu(f_n(B)) = \sum_{n=0}^{N} p_n \mu(B) = \mu(B)
\]
for every Borel set \( B \subset [0, 1] \). Hence,
\[
\varphi(x) = \mu([0, x]) = \sum_{n=0}^{N} \mu(f_n([0, x])) = \sum_{n=0}^{N} \mu([f_n(0), f_n(x)])
\]
\[
= \sum_{n=0}^{N} \mu([0, f_n(x)]) - \sum_{n=0}^{N} \mu([0, f_n(0)])
\]
\[
= \sum_{n=0}^{N} \varphi(f_n(x)) - \sum_{n=0}^{N} \varphi(f_n(0)) = \sum_{n=0}^{N} \varphi(f_n(x)) - \sum_{n=1}^{N} \varphi(f_n(0))
\]
for every \( x \in [0, 1] \).

Thus, we have proved that \( \varphi \in C \). Now the assertion of the lemma follows from Lemma 3.2; to see it the reader can consult [1, Theorem 31.7].

It is a very difficult (and still open) problem to decide for which parameters \( p_0, \ldots, p_N \) the function \( \varphi \) is absolutely continuous. However, it turns out that under some assumptions on the given contractions \( f_0, \ldots, f_N \) Eq. (E) has exactly one absolutely continuous solution in the class \( C \).

**Theorem 3.4.** Assume that \( f_0, \ldots, f_N \in C^2([0, 1]) \) and there exist \( \lambda \in (0, 1) \) and \( c \in (0, \infty) \) such that \( 0 < f'_n(x) \leq \lambda \) and \( f''_n(x) \leq cf'_n(x) \) for all \( n \in \{0, \ldots, N\} \) and \( x \in [0, 1] \). Then \( C_a \) consists of exactly one function.

**Proof.** Define \( S : [0, 1] \rightarrow [0, 1] \) by
\[
S(x) = \begin{cases} 
\frac{1}{f_n(x)} & \text{for } x \in [f_n(0), f_n(1)] \text{ and } n \in \{0, \ldots, N\}, \\
1 & \text{for } x = 1.
\end{cases}
\]

Now it is enough to apply [6, Theorem 6.2.1].

Theorem 3.4 enforces looking for these unique parameters \( p_0, \ldots, p_N \) for which \( \text{id}_{[0,1]} \in C_a \). It is still difficult in full generality. However, it can be done with success in the case where \( f_0, \ldots, f_N \) are similitudes; such a case will be considered in the next section.

Now let us set down an obvious characterization of these contractions \( f_0, \ldots, f_N \) for which \( \text{id}_{[0,1]} \in C_a \).

**Proposition 3.5.** The identity on \([0, 1]\) belongs to \( C_a \) if and only if
\[
\sum_{n=0}^{N} f_n(x) - x = \sum_{n=1}^{N} f_n(0)
\]
for every \( x \in [0, 1] \).
The last result of this section gives a precise formula for $\varphi$.

**Theorem 3.6.** Assume that $x \in [0, 1]$ and let $(x_k)_{k \in \mathbb{N}}$ be a sequence of elements of $\{0, \ldots, N\}$ such that (4) holds. Then

$$\varphi(x) = \sum_{k=1}^{\infty} \text{sgn}(x_k) \left( \prod_{n=0}^{N} p_n^{\# \{i \in \{1,\ldots,k-1\}: x_i = n\}} \cdot \sum_{n=0}^{x_k-1} p_n \right).$$

**Proof.** We begin with showing inductively that

$$\mu([f_{n_1,\ldots,n_{k-1}}(0), f_{n_1,\ldots,n_k}(0)]) = \text{sgn}(n_k) \prod_{n=0}^{N} p_n^{\# \{i \in \{1,\ldots,k-1\}: n_i = n\}} \cdot \sum_{n=0}^{n_k-1} p_n \quad (12)$$

for all $k \in \mathbb{N}$ and all $n_1, \ldots, n_k \in \{0, \ldots, N\}$.

If $n_1 = 0$, then $\text{sgn}(n_1) = 0$, and hence

$$\mu([0, f_{n_1}(0)]) = \mu([0]) = 0 = \text{sgn}(n_1) \sum_{n=0}^{n_1-1} p_n.$$

If $n_1 \geq 1$, we have $\text{sgn}(n_1) = 1$, and then by (3), (10) and Lemma 3.1 we obtain

$$\mu([0, f_{n_1}(0)]) = \sum_{n=0}^{n_1-1} \mu([f_n(0), f_n(1)]) = \sum_{n=0}^{n_1-1} p_n \mu([0, 1]) = \text{sgn}(n_1) \sum_{n=0}^{n_1-1} p_n.$$

Therefore (12) holds for $k = 1$ and all $n_1, \ldots, n_k \in \{0, \ldots, N\}$.

Fix $k \in \mathbb{N}$ and assume that (12) holds for all $n_1, \ldots, n_k \in \{0, \ldots, N\}$.

Fix $n_{k+1} \in \{0, \ldots, N\}$. Applying (10) and (12) we get

$$\mu([f_{n_1,\ldots,n_k}(0), f_{n_1,\ldots,n_{k+1}}(0)]) = p_{n_1} \mu([f_{n_2,\ldots,n_k}(0), f_{n_2,\ldots,n_{k+1}}(0)])$$

$$= p_{n_1} \text{sgn}(n_{k+1}) \prod_{n=0}^{N} p_n^{\# \{i \in \{2,\ldots,k\}: n_i = n\}} \cdot \sum_{n=0}^{n_{k+1}-1} p_n$$

$$= \text{sgn}(n_{k+1}) \prod_{n=0}^{N} p_n^{\# \{i \in \{1,2,\ldots,k\}: n_i = n\}} \cdot \sum_{n=0}^{n_{k+1}-1} p_n.$$

By the continuity of $\varphi$ (see Theorem 3.3) we have

$$\varphi(x) = \varphi \left( \lim_{t \to \infty} f_{x_1,\ldots,x_l}(0) \right) = \lim_{t \to \infty} \varphi(f_{x_1,\ldots,x_l}(0)).$$

Then using (3), Lemma 3.1 and (12) with $n_i = x_i$ for all $i \in \{1, \ldots, l\}$, we get

$$\varphi(f_{x_1,\ldots,x_l}(0)) = \mu([0, f_{x_1,\ldots,x_l}(0)]) = \sum_{k=1}^{l} \mu([f_{x_1,\ldots,x_{k-1}}(0), f_{x_1,\ldots,x_k}(0)])$$

$$= \sum_{k=1}^{l} \text{sgn}(x_k) \left( \prod_{n=0}^{N} p_n^{\# \{i \in \{1,\ldots,k-1\}: x_i = n\}} \cdot \sum_{n=0}^{x_k-1} p_n \right).$$
Passing with \( l \) to \( \infty \) we obtain the required formula for \( \varphi \). \( \square \)

4. Similitudes case

Throughout this section we assume that \( f_0, \ldots, f_N \) are similitudes, i.e. there exist real numbers \( \rho_0, \ldots, \rho_N \in (0,1) \) such that

\[
\sum_{n=0}^{N} \rho_n = 1 \tag{13}
\]

and

\[
f_n(x) = \rho_n x + \sum_{k=0}^{n-1} \rho_k
\]

for all \( x \in [0,1] \) and \( n \in \{0, \ldots, N\} \).

Note that (3) holds.

Since the above defined similitudes satisfy the assumptions of Theorem 3.4, it follows that the class \( C \) has exactly one absolutely continuous solution. Thus according to Theorem 3.3 we conclude that \( \varphi \) is singular except one very particular case of parameters \( p_0, \ldots, p_N \), which we are looking for.

**Theorem 4.1.** If \( p_n = \rho_n \) for every \( n \in \{0, \ldots, N\} \), then \( \varphi = \text{id}_{[0,1]} \).

**Proof.** Assume that \( p_n = \rho_n \) for every \( n \in \{0, \ldots, N\} \).

Observe first that applying (13), we get

\[
\sum_{n=0}^{N} f_n(x) - x = \sum_{n=0}^{N} \rho_n x + \sum_{n=0}^{N} \sum_{k=0}^{n-1} \rho_k - x = \sum_{n=0}^{N} f_n(0) = \sum_{n=1}^{N} f_n(0)
\]

for every \( x \in [0,1] \). Thus, \( \text{id}_{[0,1]} \in C_a \), by Proposition 3.5.

Now we can use Theorem 3.4 or argue as follows.

Denote by \( \nu \) the one-dimensional Lebesgue measure restricted to \([0,1]\). According to [1, Theorem 12.4] we infer that \( \nu \) is the unique Borel measure on \([0,1]\) such that \( \nu([0,x]) = x \) for every \( x \in [0,1] \). Fix \( n \in \{0, \ldots, N\} \) and choose \( x \in [f_n(0), f_n(1)] \). Then

\[
\nu([f_n(0), x]) = \nu([0, x]) - \nu([0, f_n(0)]) = x - f_n(0) = \rho_n \left( \frac{x}{\rho_n} - \sum_{k=0}^{n-1} \frac{\rho_k}{\rho_n} \right)
\]

\[
= p_n f_n^{-1}(x) = p_n \nu([0, f_n^{-1}(x)]) = p_n \nu(f_n^{-1}([f_n(0), x])).
\]

Hence

\[
\nu(A) = p_n \nu(f_n^{-1}(A))
\]
for every Borel set \( A \subset [f_n(0), f_n(1)] \), and in consequence,
\[
\nu(A) = \sum_{n=0}^{N} p_n \nu(f_n^{-1}(A))
\]
for every Borel set \( A \subset [0, 1] \). Finally, by the uniqueness of \( \mu \) we obtain
\[
\varphi(x) = \mu([0, x]) = \nu([0, x]) = x
\]
for every \( x \in [0, 1] \).

Combining Theorems 3.3, 3.4 and 4.1 we get the following corollary.

**Corollary 4.2.** If \( p_n \neq \rho_n \) for some \( n \in \{0, \ldots, N\} \), then \( \varphi \in C_s \).

Note that in our setting \( \prod_{n=0}^{N} p_n^{\rho_n} \rho_n^{-p_n} \geq 1 \). Observe also that the iterated function system consisting of the contractions \( f_0, \ldots, f_N \) satisfies the open set condition. Therefore Theorem 4.1 jointly with Corollary 4.2 can be written in the following form, which corresponds to Theorem 1.1 from [11].

**Theorem 4.3.** We have \( \varphi \in C_a \) if and only if \( p_n = \rho_n \) for every \( n \in \{0, \ldots, N\} \).

Moreover, if \( \varphi \in C_a \), then \( \varphi = \text{id}_{[0,1]} \).

To the end of this section we assume that \( \rho_0 = \rho_1 = \cdots = \rho_N = \frac{1}{N+1} \).

Note that (13) is satisfied and Eq. (E) now takes the form
\[
\varphi(x) = \sum_{n=0}^{N} \varphi \left( \frac{x + n}{N+1} \right) - \sum_{n=1}^{N} \varphi \left( \frac{n}{N+1} \right). \tag{eN}
\]
It is clear that for \( N = 1 \) Eq. (e_N) reduces to Eq. (e_1).

Fix \( x \in [0, 1] \) and define a sequence \((x_k)_{k \in \mathbb{N}}\) of elements of \( \{0, \ldots, N\} \) as follows:
- if \( x = 1 \) we put \( x_k = N \) for every \( k \in \mathbb{N} \);
- if \( x < 1 \) we put \( x_1 = \lfloor (N+1)x \rfloor \) and then inductively
  \[
  x_{k+1} = \left\lfloor (N+1)^{k+1} x - \sum_{i=1}^{k} (N+1)^{k+1-i} x_i \right\rfloor
  \]
for every \( k \in \mathbb{N} \), where \( \lfloor y \rfloor \) denotes the integer part of \( y \in \mathbb{R} \).

Clearly,
\[
x = \lim_{k \to \infty} f_{x_1, \ldots, x_k}(0) = \sum_{k=1}^{\infty} \frac{x_k}{(N+1)^k},
\]
and Theorem 3.6 yields
\[
\varphi \left( \sum_{k=1}^{\infty} \frac{x_k}{(N+1)^k} \right) = \sum_{k=1}^{\infty} \text{sgn}(x_k) \left( \prod_{n=0}^{N} p_n^{\# \{i \in \{1, 2, \ldots, k-1\} : x_i = n \}} \cdot \sum_{n=0}^{x_k-1} p_n \right). \tag{14}
\]
In particular,

$$\varphi\left(\frac{n}{N+1}\right) = \sum_{k=0}^{n-1} p_k$$  \hspace{1cm} (15)$$

for every $n \in \{1, \ldots, N\}$.

Now we are able to calculate the integral of $\varphi$ on $[0,1]$.

**Proposition 4.4.** We have

$$\int_0^1 \varphi(x) dx = \frac{1}{N} \sum_{n=1}^{N} np_{N-n}.$$  

**Proof.** Using (15) and ($e_N$), we get

$$\int_0^1 \varphi(x) dx = \sum_{n=0}^{N} \int_0^1 \varphi\left(\frac{x+n}{N+1}\right) dx - \int_0^1 \sum_{n=1}^{N} \varphi\left(\frac{n}{N+1}\right) dx$$

$$= \left(N+1\right) \sum_{n=0}^{N} \int_{\frac{n}{N+1}}^{\frac{n+1}{N+1}} \varphi(y) dy - \sum_{n=1}^{N} \varphi\left(\frac{n}{N+1}\right)$$

$$= \left(N+1\right) \int_0^1 \varphi(x) dx - \sum_{k=0}^{N-1} (N-k) p_k.$$  

This implies the required formula for the integral of $\varphi$. \hfill \Box

We end this section by observing that $\varphi$ can be extended to an increasing and continuous function satisfying ($e_N$) for every $x \in \mathbb{R}$.

**Proposition 4.5.** The function $\phi: \mathbb{R} \to \mathbb{R}$ given by

$$\phi(x) = \lfloor x \rfloor + \varphi(x - \lfloor x \rfloor)$$

is increasing, continuous and satisfies ($e_N$) for every $x \in \mathbb{R}$.

**Proof.** Fix $x \in \mathbb{R}$ and assume that $x \in [m(N+1)+l, m(N+1)+l+1)$ for some $m \in \mathbb{Z}$ and $l \in \{0, 1, \ldots, N\}$. Then $\lfloor \frac{x+i}{N+1} \rfloor = m$ for every $i \in \{0, \ldots, N-l\}$ and $\lfloor \frac{x+i}{N+1} \rfloor = m+1$ for every $i \in \{N-l+1, \ldots, N\}$. Consequently,
\[
\phi(x) = [x] + \varphi(x - [x]) = \\
= m(N + 1) + l + \sum_{n=0}^{N} \varphi \left( \frac{x - [x] + n}{N + 1} \right) - \sum_{n=1}^{N} \varphi \left( \frac{n}{N + 1} \right) \\
= m(N + 1) + l + \sum_{n=0}^{N} \varphi \left( \frac{x + n - l}{N + 1} - m \right) - \sum_{n=1}^{N} \varphi \left( \frac{n}{N + 1} \right) \\
= (m + 1)l + \sum_{n=0}^{l-1} \varphi \left( \frac{x + n - l + N + 1}{N + 1} - m - 1 \right) \\
+ m(N + 1 - l) + \sum_{n=l}^{N} \varphi \left( \frac{x + n - l}{N + 1} - m \right) - \sum_{n=1}^{N} \varphi \left( \frac{n}{N + 1} \right) \\
= (m + 1)l + \sum_{n=N+1-l}^{N} \varphi \left( \frac{x + n - m - 1}{N + 1} \right) \\
+ m(N - l + 1) + \sum_{n=0}^{N-1} \varphi \left( \frac{x + n - m}{N + 1} \right) - \sum_{n=1}^{N} \varphi \left( \frac{n}{N + 1} \right) \\
= \sum_{n=N-l+1}^{N} \left\{ \left[ \frac{x + n}{N + 1} \right] + \varphi \left( \frac{x + n}{N + 1} - \left[ \frac{x + n}{N + 1} \right] \right) \right\} \\
+ \sum_{n=0}^{N-1} \left\{ \left[ \frac{x + n}{N + 1} \right] + \varphi \left( \frac{x + n}{N + 1} - \left[ \frac{x + n}{N + 1} \right] \right) \right\} - \sum_{n=1}^{N} \varphi \left( \frac{n}{N + 1} \right) \\
= \sum_{n=0}^{N} \phi \left( \frac{x + n}{N + 1} \right) - \sum_{n=1}^{N} \phi \left( \frac{n}{N + 1} \right).
\]

To prove that \( \phi \) is increasing fix \( x < y \). If \( [x] = [y] \), then

\[
\phi(x) = [x] + \varphi(x - [x]) = [y] + \varphi(x - [y]) \leq [y] + \varphi(y - [y]) = \phi(y),
\]

and if \( [x] < [y] \), then

\[
\phi(x) = [x] + \varphi(x - [x]) \leq [y] \leq [y] + \varphi(y - [y]) = \phi(y).
\]

It is clear that \( \phi \) is continuous at every point of the set \( \mathbb{R} \setminus \mathbb{Z} \). If \( k \in \mathbb{Z} \), then by the continuity of \( \varphi \) and (1) we obtain

\[
\lim_{x \to k^+} \phi(x) = \lim_{x \to k^+} \left( [x] + \varphi(x - [x]) \right) = k + \lim_{y \to 0^+} \varphi(y) = k = \phi(k)
\]

and

\[
\lim_{x \to k^-} \phi(x) = \lim_{x \to k^-} \left( [x] + \varphi(x - [x]) \right) = k - 1 + \lim_{y \to 1^-} \varphi(y) = k,
\]

which completes the proof. \qed
5. Matkowski–Wesołowski case

First of all observe that formula (14) with $N = 1$ coincides with formula (2). So the main part of assertion (ii) of Theorem 1.1 is a very special case of Theorem 3.6, whereas its moreover part follows from Corollary 4.2. Now we would like to get a little bit more information about the class $C$. For this purpose, we denote the convex hull of a set $A$ by $\text{conv}(A)$ and put

$$W = \{\varphi_p : p \in (0, 1)\},$$

where $\varphi_p : [0, 1] \to [0, 1]$ is the function defined by (2).

**Proposition 5.1.** The set $W$ is linearly independent. Moreover:

(i) $\text{conv}(W) \subset C$;
(ii) $\text{conv}(W\setminus\{\varphi_{\frac{1}{2}}\}) \subset C_s$.

**Proof.** To prove that $W$ is linearly independent fix $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $0 < p_1 < p_2 < \cdots < p_n < 1$ and assume that

$$\sum_{i=1}^{n} \alpha_i \varphi_{p_i}(x) = 0,$$

for every $x \in [0, 1]$. Applying (2) we conclude that $\varphi_{p_i}(\frac{1}{2^k}) = p_i^k$ for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$. Then for every $k \in \mathbb{N}$ we have

$$\sum_{i=1}^{n} \alpha_i \left(\frac{p_i}{p_n}\right)^k = 0.$$

Taking the limit as $k \to \infty$ we get $\alpha_n = 0$. Repeating this procedure $n - 1$ times gives $\alpha_n = \alpha_{n-1} = \cdots = \alpha_1 = 0$.

Assertion (i) follows from Remark 2.1 and assertion (ii) is a consequence of the moreover part of assertion (ii) of Theorem 1.1. \qed

To formulate an answer to the problem posed in [9] by Janusz Matkowski define first a function $\varphi_1 : [0, 1] \to \mathbb{R}$ putting $\varphi_1(x) = 1$ and observe that by Proposition 5.1 and the fact that $\varphi_1(0) = 1$ and $\varphi_p(0) = 0$ for every $p \in (0, 1)$ the set $W\cup\{\varphi_1\}$ is linearly independent. Let $\mathcal{M}$ denote the vector space whose basis is $W\cup\{\varphi_1\}$, i.e.

$$\mathcal{M} = \text{lin}(W\cup\{\varphi_1\}).$$

Applying Proposition 5.1 and Remark 1.2, we get the following result.

**Theorem 5.2.** Every function belonging to $\mathcal{M}$ is a continuous solution of Eq. (e_1). Moreover, $\sum_{i=1}^{n} \alpha_i \varphi_{p_i} \in \mathcal{M}$ is:

(i) monotone provided that $\text{sgn}(\alpha_i) = \text{sgn}(\alpha_j)$ for all $i, j \in \{1, \ldots, n\}$ such that $p_i, p_j \in (0, 1)$;
(ii) singular for all $p_1, \ldots, p_n \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. 
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Janusz Morawiec and Thomas Zürcher
Instytut Matematyki
Uniwersytet Śląski
Bankowa 14
40-007 Katowice
Poland
e-mail: morawiec@math.us.edu.pl

Thomas Zürcher
Mathematics Institute
University of Warwick
Coventry CV4 7AL
UK
e-mail: thomas.zurcher@us.edu.pl;
   T.Zurcher@warwick.ac.uk

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