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On a problem of Janusz Matkowski and Jacek Wesołowski

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Abstract. We study the problem of the existence of increasing and continuous solutions $\varphi\colon [0,1] \to [0,1]$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$ of the functional equation

$$\varphi(x) = \sum_{n=0}^{N} \varphi(f_n(x)) - \sum_{n=1}^{N} \varphi(f_n(0)),$$

where $N \in \mathbb{N}$ and $f_0, \ldots, f_N\colon [0,1] \to [0,1]$ are strictly increasing contractions satisfying the following condition $0 = f_0(0) < f_0(1) = f_1(0) < \cdots < f_{N-1}(1) = f_N(0) < f_N(1) = 1$. In particular, we give an answer to the problem posed in Matkowski (Aequationes Math. 29:210–213, 1985) by Janusz Matkowski concerning a very special case of that equation.

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1. Introduction

During the 47th International Symposium on Functional Equations in 2009 Jacek Wesołowski asked whether the identity on $[0,1]$ is the only increasing and continuous solution $\varphi\colon [0,1] \to [0,1]$ of the equation

$$\varphi(x) = \varphi\left(\frac{x}{2}\right) + \varphi\left(\frac{x+1}{2}\right) - \varphi\left(\frac{1}{2}\right) \tag{e_1}$$

satisfying

$$\varphi(0) = 0 \qquad \text{and} \qquad \varphi(1) = 1. \tag{1}$$

This question was posed in connection with studying probability measures in the plane which are invariant by "winding" (see [10]).

A negative answer to this question was obtained in [5] and reads as follows.

Theorem 1.1. (i) *The identity on* $[0,1]$ *is the only increasing and absolutely continuous solution* $\varphi\colon [0,1] \to [0,1]$ *of Eq.* (e_1) *satisfying* (1).

(ii) *For every* $p \in (0,1)$ *the function* $\varphi_p\colon [0,1] \to [0,1]$ *given by*

$$\varphi_p\left(\sum_{k=1}^{\infty} \frac{x_k}{2^k}\right) = \sum_{k=1}^{\infty} x_k p^{k-\sum_{i=1}^{k-1} x_i}(1-p)^{\sum_{i=1}^{k-1} x_i}, \tag{2}$$

where $x_k \in \{0,1\}$ *for all* $k \in \mathbb{N}$*, is an increasing and continuous solution of Eq.* (e_1) *satisfying* (1)*. Moreover,* φ_p *is singular for every* $p \neq \frac{1}{2}$*.*

Let us note that the first assertion of Theorem 1.1 is known (see e.g. [13] or [8]), however in [5] we can find an independent proof of it.

It turns out that in 1985 Janusz Matkowski posed a problem asking if Eq. (e_1) has a non-linear monotonic and continuous solution $\varphi\colon [0,1] \to \mathbb{R}$ (see [9]). Moreover, he observed that monotonic solutions of Eq. (e_1) are connected with invariant measures for a certain map on $[0,1]$. Note that Matkowski's problem is equivalent to Wesołowski's question.

Remark 1.2. (i) If $\varphi\colon [0,1] \to [0,1]$ is an increasing and continuous solution of Eq. (e_1) satisfying (1), then for all $a, b \in \mathbb{R}$ the function $a\varphi + b$ is monotonic, continuous and satisfies (e_1) for every $x \in [0,1]$.

(ii) If $\varphi\colon [0,1] \to \mathbb{R}$ is a monotonic and continuous solution of Eq. (e_1), different from a constant function, then $\frac{\varphi-\varphi(0)}{\varphi(1)-\varphi(0)}$ is an increasing and continuous function satisfying (1) and (e_1) for every $x \in [0,1]$.

2. Preliminaries

Fix $N \in \mathbb{N}$, strictly increasing contractions $f_0, \ldots, f_N\colon [0,1] \to [0,1]$ such that

$$0 = f_0(0) < f_0(1) = f_1(0) < \cdots < f_{N-1}(1) = f_N(0) < f_N(1) = 1 \tag{3}$$

and consider the functional equation

$$\varphi(x) = \sum_{n=0}^{N} \varphi(f_n(x)) - \sum_{n=1}^{N} \varphi(f_n(0)) \tag{E}$$

for every $x \in [0,1]$. Denote by $\mathcal{C}$ the class of all continuous and increasing solutions $\varphi\colon [0,1] \to [0,1]$ of Eq. (E) satisfying (1). Following the idea from [5] we show that $\mathcal{C}$ contains many functions, however, we manage to identify a quite large class of contractions that includes the similitudes such that there is exactly one absolutely continuous solution.

We begin with two observations showing that in many situations the class $\mathcal{C}$ is determined by two of its subclasses $\mathcal{C}_a$ and $\mathcal{C}_s$, consisting of all absolutely continuous and all singular functions, respectively.

Remark 2.1. If $\varphi_1, \varphi_2 \in \mathcal{C}$ and if $\alpha \in (0,1)$, then $\alpha\varphi_1 + (1-\alpha)\varphi_2 \in \mathcal{C}$.

To formulate the next remark we recall that a Lebesgue measurable function $f\colon [0,1] \to [0,1]$ is said to be *nonsingular* if the set $f^{-1}(A)$ has Lebesgue measure zero for every set $A \subset [0,1]$ of Lebesgue measure zero (see [6]). Observe that an invertible Lebesgue measurable function f is nonsingular if and only if its inverse f^{-1} satisfies Luzin's condition (N).

Remark 2.2. Assume that all the contractions $f_0, \ldots, f_N$ are nonsingular. Then, both the absolutely continuous and the singular parts of every element from $\mathcal{C}$ satisfy (E) for every $x \in [0,1]$.

Proof. Fix $\varphi \in \mathcal{C}$ and denote by φ_a and φ_s its absolutely continuous and singular parts,[1] respectively. By (E), for every $x \in [0,1]$ we have

$$\varphi_a(x) - \sum_{n=0}^{N} \varphi_a(f_n(x)) = -\varphi_s(x) + \sum_{n=0}^{N} \varphi_s(f_n(x)) - \sum_{n=1}^{N} \varphi(f_n(0)),$$

and hence there exists a real constant c such that

$$\varphi_a(x) - \sum_{n=0}^{N} \varphi_a(f_n(x)) = c \quad \text{and} \quad -\varphi_s(x) + \sum_{n=0}^{N} \varphi_s(f_n(x)) - \sum_{n=1}^{N} \varphi(f_n(0)) = c.$$

This jointly with the fact that $f_0(0) = 0$ stipulated in (3) gives

$$c = \varphi_a(0) - \sum_{n=0}^{N} \varphi_a(f_n(0)) = -\sum_{n=1}^{N} \varphi_a(f_n(0)),$$

and in consequence

$$\varphi_a(x) = \sum_{n=0}^{N} \varphi_a(f_n(x)) - \sum_{n=1}^{N} \varphi_a(f_n(0))$$

and

$$\varphi_s(x) = \sum_{n=0}^{N} \varphi_s(f_n(x)) - \sum_{n=1}^{N} \varphi_s(f_n(0))$$

for every $x \in [0,1]$. □

For all $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \{0, \ldots, N\}$ denote by $f_{n_1,\ldots,n_k}$ the composition $f_{n_1} \circ \cdots \circ f_{n_k}$. We extend the notation to the case $k = 0$ by letting $f_{n_1,\ldots,n_0}$ be the identity.

Lemma 2.3. *Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of elements of $\{0, \ldots, N\}$. Then the sequence $(f_{n_1,\ldots,n_k}(0))_{k\in\mathbb{N}}$ is increasing and the sequence $(f_{n_1,\ldots,n_k}(1))_{k\in\mathbb{N}}$ is decreasing. Moreover,*

$$\lim_{k\to\infty} f_{n_1,\ldots,n_k}(y) = \lim_{k\to\infty} f_{n_1,\ldots,n_k}(z)$$

[1] The parts are unique up to a constant. For definiteness, we choose them such that $\varphi_a(0) = \varphi_s(0) = 0$.

for all $y, z \in [0,1]$.

Proof. Fix a sequence $(n_k)_{k\in\mathbb{N}}$ of elements of $\{0,\dots,N\}$ and an integer number $k \geq 2$. From (3) we have

$$0 \leq f_{n_k}(0) < f_{n_k}(1) \leq 1$$

and by the strict monotonicity of $f_{n_1,\dots,n_{k-1}}$ we conclude that

$$f_{n_1,\dots,n_{k-1}}(0) \leq f_{n_1,\dots,n_k}(0) < f_{n_1,\dots,n_k}(1) \leq f_{n_1,\dots,n_{k-1}}(1).$$

To complete the proof it is enough to observe that for all $y, z \in [0,1]$ and $k \in \mathbb{N}$ we have

$$|f_{n_1,\dots,n_k}(y) - f_{n_1,\dots,n_k}(z)| \leq f_{n_1,\dots,n_k}(1) - f_{n_1,\dots,n_k}(0) \leq c^k,$$

where $c \in (0,1)$ is the largest Lipschitz constant of the given contractions $f_0,\dots,f_N$. □

Lemma 2.4. *For every* $x \in [0,1]$ *there exists a sequence* $(x_k)_{k\in\mathbb{N}}$ *of elements of* $\{0,\dots,N\}$ *such that*

$$x = \lim_{k\to\infty} f_{x_1,\dots,x_k}(0). \tag{4}$$

Proof. Fix $x \in [0,1]$ and observe that according to Lemma 2.3 it is enough to show that there exists a sequence $(x_k)_{k\in\mathbb{N}}$ of elements of $\{0,\dots,N\}$ such that

$$f_{x_1,\dots,x_k}(0) \leq x \leq f_{x_1,\dots,x_k}(1) \tag{5}$$

for every $k \in \mathbb{N}$.

By (3) there exists $x_1 \in \{0,\dots,N\}$ such that

$$f_{x_1}(0) \leq x \leq f_{x_1}(1).$$

Thus, (5) holds for $k = 1$.

Fix $k \in \mathbb{N}$ and assume inductively that there exist $x_1,\dots,x_k \in \{0,\dots,N\}$ such that (5) holds. Then

$$0 \leq f^{-1}_{x_1,\dots,x_k}(x) \leq 1$$

and by (3) there exists $x_{k+1} \in \{0,\dots,N\}$ such that

$$f_{x_{k+1}}(0) \leq f^{-1}_{x_1,\dots,x_k}(x) \leq f_{x_{k+1}}(1).$$

Hence

$$f_{x_1,\dots,x_{k+1}}(0) \leq x \leq f_{x_1,\dots,x_{k+1}}(1),$$

and the proof is complete. □

3. General case

Fix positive real numbers $p_0, \ldots, p_N$ such that

$$\sum_{n=0}^{N} p_n = 1. \tag{6}$$

Then there exists a unique Borel probability measure μ such that

$$\mu(A) = \sum_{n=0}^{N} p_n \mu(f_n^{-1}(A)) \tag{7}$$

for every Borel set $A \subset [0,1]$ (see [4]; cf. [3]). From now on the letter μ will be reserved for the unique Borel probability measure satisfying (7) for every Borel set $A \subset [0,1]$.

Lemma 3.1. *The measure μ is continuous.*

Proof. As a first step we want to show that

$$\mu\big(\{f_n(0)\}\big) = \mu\big(\{f_n(1)\}\big) = 0 \tag{8}$$

for every $n \in \{0, \ldots, N\}$.

Applying (7) and using (3), we obtain

$$\mu(\{0\}) = \mu(\{f_0(0)\}) = \sum_{n=0}^{N} p_n \mu(\{f_n^{-1}(f_0(0))\}) = p_0 \mu(\{0\}) + \sum_{n=1}^{N} p_n \mu(\emptyset).$$

By the fact that $p_0 \in (0,1)$ we conclude that

$$\mu(\{f_0(0)\}) = \mu(\{0\}) = 0.$$

Similarly, applying (7), (3) and the fact that $p_N \in (0,1)$ we conclude that

$$\mu(\{f_N(1)\}) = \mu(\{1\}) = 0.$$

If $n \in \{1, \ldots, N\}$, then applying again (7) and (3), we obtain

$$\mu(\{f_{n-1}(1)\}) = \mu(\{f_n(0)\}) = p_{n-1}\mu(\{1\}) + p_n \mu(\{0\}) = 0.$$

Our second step is to prove that

$$\mu\big([f_{n_1,\ldots,n_k}(0), f_{n_1,\ldots,n_k}(1)]\big) = \prod_{n=0}^{N} p_n^{\#\{i \in \{1,\ldots,k\} : n_i = n\}} \tag{9}$$

for all $k \in \mathbb{N} \cup \{0\}$ and $n_1, \ldots, n_k \in \{0, \ldots, N\}$.

Since $\mu([0,1]) = 1$, it follows that (9) is satisfied for $k = 0$.

Fix $k \in \mathbb{N} \cup \{0\}$ and assume that (9) holds for all $n_1, \ldots, n_k \in \{0, \ldots, N\}$. Fix also $n_{k+1} \in \{0, \ldots, N\}$.

Note first that from (8), (3), and (7), we get

$$\mu(B) = p_n \mu(f_n^{-1}(B)) \tag{10}$$

for all $n \in \{0, \dots, N\}$ and Borel sets $B \subset [f_n(0), f_n(1)]$. This jointly with (9) implies

$$\begin{aligned}\mu\big([f_{n_1,\dots,n_{k+1}}(0), f_{n_1,\dots,n_{k+1}}(1)]\big) &= p_{n_1}\mu\big([f_{n_2,\dots,n_{k+1}}(0), f_{n_2,\dots,n_{k+1}}(1)]\big)\\ &= p_{n_1}\prod_{n=0}^{N} p_n^{\#\{i\in\{2,\dots,k+1\}:n_i=n\}}\\ &= \prod_{n=0}^{N} p_n^{\#\{i\in\{1,\dots,k+1\}:n_i=n\}}.\end{aligned}$$

To prove that μ is continuous it is sufficient to show that μ has no atoms.

Fix $x \in [0,1]$. From Lemma 2.4 we conclude that there exists a sequence $(x_k)_{k\in\mathbb{N}}$ of elements of $\{0, \dots, N\}$ such that (4) holds. Then applying Lemma 2.3 and (9) with $n_i = x_i$ for $i \in \{1, \dots, k\}$, we obtain

$$\begin{aligned}\mu(\{x\}) &= \mu\left(\bigcap_{k\in\mathbb{N}}\big[f_{x_1,\dots,x_k}(0), f_{x_1,\dots,x_k}(1)\big]\right)\\ &= \lim_{k\to\infty}\mu\left(\big[f_{x_1,\dots,x_k}(0), f_{x_1,\dots,x_k}(1)\big]\right)\\ &= \lim_{k\to\infty}\prod_{n=0}^{N} p_n^{\#\{i\in\{1,\dots,k\}:x_i=n\}} \le \lim_{k\to\infty}\left(\max\{p_0,\dots,p_N\}\right)^k = 0,\end{aligned}$$

and the proof is complete. □

The next lemma is folklore (the reader can consult [2,12] in the case where $f_0, \dots, f_N$ are similitudes and [7] in the case where $f_0, \dots, f_N$ are contractions). More general results in this direction can be found e.g. in [14,15].

Lemma 3.2. *The measure μ is either singular or absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Define the function $\varphi\colon [0,1] \to [0,1]$ by

$$\varphi(x) = \mu([0,x]).$$

From now on the letter φ will be reserved for the just defined function.

Theorem 3.3. *Either $\varphi \in \mathcal{C}_a$ or $\varphi \in \mathcal{C}_s$.*

Proof. We first prove that $\varphi \in \mathcal{C}$.

That φ is increasing is a consequence of the monotonicity of μ. The continuity of φ and that $\varphi(0) = 0$ follows from Lemma 3.1. Since μ is a probability measure, we have $\varphi(1) = 1$.

From (10) we get

$$\mu(f_n(B)) = p_n\mu(B)$$

for all $n \in \{0, \dots, N\}$ and Borel sets $B \subset [0,1]$. This jointly with (6) gives

$$\sum_{n=0}^{N} \mu(f_n(B)) = \sum_{n=0}^{N} p_n \mu(B) = \mu(B)$$

for every Borel set $B \subset [0,1]$. Hence,

$$\begin{aligned}\varphi(x) = \mu([0,x]) &= \sum_{n=0}^{N} \mu\big(f_n([0,x])\big) = \sum_{n=0}^{N} \mu\big([f_n(0), f_n(x)]\big) \\ &= \sum_{n=0}^{N} \mu\big([0, f_n(x)]\big) - \sum_{n=0}^{N} \mu\big([0, f_n(0)]\big) \\ &= \sum_{n=0}^{N} \varphi(f_n(x)) - \sum_{n=0}^{N} \varphi(f_n(0)) = \sum_{n=0}^{N} \varphi(f_n(x)) - \sum_{n=1}^{N} \varphi(f_n(0))\end{aligned}$$

for every $x \in [0,1]$.

Thus, we have proved that $\varphi \in \mathcal{C}$. Now the assertion of the lemma follows from Lemma 3.2; to see it the reader can consult [1, Theorem 31.7]. □

It is a very difficult (and still open) problem to decide for which parameters $p_0, \dots, p_N$ the function φ is absolutely continuous. However, it turns out that under some assumptions on the given contractions $f_0, \dots, f_N$ Eq. (E) has exactly one absolutely continuous solution in the class $\mathcal{C}$.

Theorem 3.4. *Assume that $f_0, \dots, f_N \in C^2([0,1])$ and there exist $\lambda \in (0,1)$ and $c \in (0,\infty)$ such that $0 < f_n'(x) \le \lambda$ and $f_n''(x) \le c f_n'(x)$ for all $n \in \{0, \dots, N\}$ and $x \in [0,1]$. Then $\mathcal{C}_a$ consists of exactly one function.*

Proof. Define $S\colon [0,1] \to [0,1]$ by

$$S(x) = \begin{cases} f_n^{-1}(x) & \text{for } x \in [f_n(0), f_n(1)) \text{ and } n \in \{0, \dots, N\}, \\ 1 & \text{for } x = 1. \end{cases}$$

Now it is enough to apply [6, Theorem 6.2.1]. □

Theorem 3.4 enforces looking for these unique parameters $p_0, \dots, p_N$ for which $\varphi \in \mathcal{C}_a$. It is still difficult in full generality. However, it can be done with success in the case where $f_0, \dots, f_N$ are similitudes; such a case will be considered in the next section.

Now let us set down an obvious characterization of these contractions $f_0, \dots, f_N$ for which $\mathrm{id}_{[0,1]} \in \mathcal{C}_a$.

Proposition 3.5. *The identity on $[0,1]$ belongs to $\mathcal{C}_a$ if and only if*

$$\sum_{n=0}^{N} f_n(x) - x = \sum_{n=1}^{N} f_n(0) \tag{11}$$

for every $x \in [0,1]$.

The last result of this section gives a precise formula for φ.

Theorem 3.6. *Assume that $x \in [0,1]$ and let $(x_k)_{k\in\mathbb{N}}$ be a sequence of elements of $\{0,\dots,N\}$ such that (4) holds. Then*

$$\varphi(x) = \sum_{k=1}^{\infty} \operatorname{sgn}(x_k) \left(\prod_{n=0}^{N} p_n^{\#\{i\in\{1,\dots,k-1\}:x_i=n\}} \cdot \sum_{n=0}^{x_k-1} p_n \right).$$

Proof. We begin with showing inductively that

$$\mu\big([f_{n_1,\dots,n_{k-1}}(0), f_{n_1,\dots,n_k}(0)]\big) = \operatorname{sgn}(n_k) \prod_{n=0}^{N} p_n^{\#\{i\in\{1,\dots,k-1\}:n_i=n\}} \cdot \sum_{n=0}^{n_k-1} p_n \tag{12}$$

for all $k \in \mathbb{N}$ and all $n_1,\dots,n_k \in \{0,\dots,N\}$.

If $n_1 = 0$, then $\operatorname{sgn}(n_1) = 0$, and hence

$$\mu\big([0, f_{n_1}(0)]\big) = \mu\big(\{0\}\big) = 0 = \operatorname{sgn}(n_1) \sum_{n=0}^{n_1-1} p_n.$$

If $n_1 \geq 1$, we have $\operatorname{sgn}(n_1) = 1$, and then by (3), (10) and Lemma 3.1 we obtain

$$\mu\big([0, f_{n_1}(0)]\big) = \sum_{n=0}^{n_1-1} \mu\big([f_n(0), f_n(1)]\big) = \sum_{n=0}^{n_1-1} p_n \mu([0,1]) = \operatorname{sgn}(n_1) \sum_{n=0}^{n_1-1} p_n.$$

Therefore (12) holds for $k = 1$ and all $n_1,\dots,n_k \in \{0,\dots,N\}$.

Fix $k \in \mathbb{N}$ and assume that (12) holds for all $n_1,\dots,n_k \in \{0,\dots,N\}$.

Fix $n_{k+1} \in \{0,\dots,N\}$. Applying (10) and (12) we get

$$\begin{aligned}
\mu\big([f_{n_1,\dots,n_k}(0), f_{n_1,\dots,n_{k+1}}(0)]\big) &= p_{n_1} \mu\big([f_{n_2,\dots,n_k}(0), f_{n_2,\dots,n_{k+1}}(0)]\big) \\
&= p_{n_1} \operatorname{sgn}(n_{k+1}) \prod_{n=0}^{N} p_n^{\#\{i\in\{2,\dots,k\}:n_i=n\}} \cdot \sum_{n=0}^{n_{k+1}-1} p_n \\
&= \operatorname{sgn}(n_{k+1}) \prod_{n=0}^{N} p_n^{\#\{i\in\{1,2,\dots,k\}:n_i=n\}} \cdot \sum_{n=0}^{n_{k+1}-1} p_n.
\end{aligned}$$

By the continuity of φ (see Theorem 3.3) we have

$$\varphi(x) = \varphi\left(\lim_{l\to\infty} f_{x_1,\dots,x_l}(0) \right) = \lim_{l\to\infty} \varphi(f_{x_1,\dots,x_l}(0)).$$

Then using (3), Lemma 3.1 and (12) with $n_i = x_i$ for all $i \in \{1,\dots l\}$, we get

$$\begin{aligned}
\varphi(f_{x_1,\dots,x_l}(0)) = \mu([0, f_{x_1,\dots,x_l}(0)]) &= \sum_{k=1}^{l} \mu([f_{x_1,\dots,x_{k-1}}(0), f_{x_1,\dots,x_k}(0)]) \\
&= \sum_{k=1}^{l} \operatorname{sgn}(x_k) \left(\prod_{n=0}^{N} p_n^{\#\{i\in\{1,\dots,k-1\}:x_i=n\}} \cdot \sum_{n=0}^{x_k-1} p_n \right).
\end{aligned}$$

Passing with l to ∞ we obtain the required formula for φ. $\square$

4. Similitudes case

Throughout this section we assume that $f_0, \ldots, f_N$ are similitudes, i.e. there exist real numbers $\rho_0, \ldots, \rho_N \in (0,1)$ such that

$$\sum_{n=0}^{N} \rho_n = 1 \tag{13}$$

and

$$f_n(x) = \rho_n x + \sum_{k=0}^{n-1} \rho_k$$

for all $x \in [0,1]$ and $n \in \{0, \ldots, N\}$.

Note that (3) holds.

Since the above defined similitudes satisfy the assumptions of Theorem 3.4, it follows that the class $\mathcal{C}$ has exactly one absolutely continuous solution. Thus according to Theorem 3.3 we conclude that φ is singular except one very particular case of parameters $p_0, \ldots, p_N$, which we are looking for.

Theorem 4.1. *If $p_n = \rho_n$ for every $n \in \{0, \ldots, N\}$, then $\varphi = \mathrm{id}_{[0,1]}$.*

Proof. Assume that $p_n = \rho_n$ for every $n \in \{0, \ldots, N\}$.

Observe first that applying (13), we get

$$\sum_{n=0}^{N} f_n(x) - x = \sum_{n=0}^{N} \rho_n x + \sum_{n=0}^{N} \sum_{k=0}^{n-1} \rho_k - x = \sum_{n=0}^{N} f_n(0) = \sum_{n=1}^{N} f_n(0)$$

for every $x \in [0,1]$. Thus, $\mathrm{id}_{[0,1]} \in \mathcal{C}_a$, by Proposition 3.5.

Now we can use Theorem 3.4 or argue as follows.

Denote by ν the one-dimensional Lebesgue measure restricted to $[0,1]$. According to [1, Theorem 12.4] we infer that ν is the unique Borel measure on $[0,1]$ such that $\nu([0,x]) = x$ for every $x \in [0,1]$. Fix $n \in \{0, \ldots, N\}$ and choose $x \in \big[f_n(0), f_n(1)\big]$. Then

$$\begin{aligned}\nu([f_n(0), x]) &= \nu([0,x]) - \nu([0, f_n(0)]) = x - f_n(0) = \rho_n \left(\frac{x}{\rho_n} - \sum_{k=0}^{n-1} \frac{\rho_k}{\rho_n} \right) \\ &= p_n f_n^{-1}(x) = p_n \nu\big([0, f_n^{-1}(x)]\big) = p_n \nu\big(f_n^{-1}([f_n(0), x])\big).\end{aligned}$$

Hence

$$\nu(A) = p_n \nu(f_n^{-1}(A))$$

for every Borel set $A \subset [f_n(0), f_n(1)]$, and in consequence,

$$\nu(A) = \sum_{n=0}^{N} p_n \nu(f_n^{-1}(A))$$

for every Borel set $A \subset [0,1]$. Finally, by the uniqueness of μ we obtain

$$\varphi(x) = \mu([0,x]) = \nu([0,x]) = x$$

for every $x \in [0,1]$. □

Combining Theorems 3.3, 3.4 and 4.1 we get the following corollary.

Corollary 4.2. *If $p_n \neq \rho_n$ for some $n \in \{0,\dots,N\}$, then $\varphi \in \mathcal{C}_s$.*

Note that in our setting $\prod_{n=0}^{N} p_n^{p_n} \rho_n^{-p_n} \geq 1$. Observe also that the iterated function system consisting of the contractions $f_0,\dots,f_N$ satisfies the open set condition. Therefore Theorem 4.1 jointly with Corollary 4.2 can be written in the following form, which corresponds to Theorem 1.1 from [11].

Theorem 4.3. *We have $\varphi \in \mathcal{C}_a$ if and only if $p_n = \rho_n$ for every $n \in \{0,\dots,N\}$. Moreover, if $\varphi \in \mathcal{C}_a$, then $\varphi = \mathrm{id}_{[0,1]}$.*

To the end of this section we assume that

$$\rho_0 = \rho_1 = \dots = \rho_N = \frac{1}{N+1}.$$

Note that (13) is satisfied and Eq. (E) now takes the form

$$\varphi(x) = \sum_{n=0}^{N} \varphi\left(\frac{x+n}{N+1}\right) - \sum_{n=1}^{N} \varphi\left(\frac{n}{N+1}\right). \tag{e_N}$$

It is clear that for $N = 1$ Eq. (e_N) reduces to Eq. (e_1).

Fix $x \in [0,1]$ and define a sequence $(x_k)_{k\in\mathbb{N}}$ of elements of $\{0,\dots,N\}$ as follows:

if $x = 1$ we put $x_k = N$ for every $k \in \mathbb{N}$;

if $x < 1$ we put $x_1 = [(N+1)x]$ and then inductively

$$x_{k+1} = \left[(N+1)^{k+1}x - \sum_{i=1}^{k} (N+1)^{k+1-i} x_i\right]$$

for every $k \in \mathbb{N}$, where $[y]$ denotes the integer part of $y \in \mathbb{R}$.

Clearly,

$$x = \lim_{k\to\infty} f_{x_1,\dots,x_k}(0) = \sum_{k=1}^{\infty} \frac{x_k}{(N+1)^k},$$

and Theorem 3.6 yields

$$\varphi\left(\sum_{k=1}^{\infty} \frac{x_k}{(N+1)^k}\right) = \sum_{k=1}^{\infty} \mathrm{sgn}(x_k) \left(\prod_{n=0}^{N} p_n^{\#\{i\in\{1,2,\dots,k-1\}:x_i=n\}} \cdot \sum_{n=0}^{x_k-1} p_n\right). \tag{14}$$

In particular,

$$\varphi\left(\frac{n}{N+1}\right) = \sum_{k=0}^{n-1} p_k \tag{15}$$

for every $n \in \{1, \ldots, N\}$.

Now we are able to calculate the integral of φ on $[0, 1]$.

Proposition 4.4. *We have*

$$\int_0^1 \varphi(x)dx = \frac{1}{N}\sum_{n=1}^{N} np_{N-n}.$$

Proof. Using (15) and (e_N), we get

$$\begin{aligned}\int_0^1 \varphi(x)dx &= \sum_{n=0}^{N}\int_0^1 \varphi\left(\frac{x+n}{N+1}\right)dx - \int_0^1 \sum_{n=1}^{N}\varphi\left(\frac{n}{N+1}\right)dx \\ &= (N+1)\sum_{n=0}^{N}\int_{\frac{n}{N+1}}^{\frac{n+1}{N+1}} \varphi\left(y\right)dy - \sum_{n=1}^{N}\varphi\left(\frac{n}{N+1}\right) \\ &= (N+1)\int_0^1 \varphi(x)\,dx - \sum_{k=0}^{N-1}(N-k)p_k.\end{aligned}$$

This implies the required formula for the integral of φ. □

We end this section by observing that φ can be extended to an increasing and continuous function satisfying (e_N) for every $x \in \mathbb{R}$.

Proposition 4.5. *The function* $\phi\colon \mathbb{R} \to \mathbb{R}$ *given by*

$$\phi(x) = [x] + \varphi(x - [x])$$

is increasing, continuous and satisfies (e_N) *for every* $x \in \mathbb{R}$.

Proof. Fix $x \in \mathbb{R}$ and assume that $x \in [m(N+1)+l, m(N+1)+l+1)$ for some $m \in \mathbb{Z}$ and $l \in \{0, 1, \ldots, N\}$. Then $[\frac{x+i}{N+1}] = m$ for every $i \in \{0, \ldots, N-l\}$ and $[\frac{x+i}{N+1}] = m+1$ for every $i \in \{N-l+1, \ldots, N\}$. Consequently,

$$
\begin{aligned}
\phi(x) &= [x] + \varphi(x - [x]) = \\
&= m(N+1) + l + \sum_{n=0}^{N} \varphi\left(\frac{x - [x] + n}{N+1}\right) - \sum_{n=1}^{N} \varphi\left(\frac{n}{N+1}\right) \\
&= m(N+1) + l + \sum_{n=0}^{N} \varphi\left(\frac{x + n - l}{N+1} - m\right) - \sum_{n=1}^{N} \varphi\left(\frac{n}{N+1}\right) \\
&= (m+1)l + \sum_{n=0}^{l-1} \varphi\left(\frac{x + n - l + N + 1}{N+1} - m - 1\right) \\
&\quad + m(N+1-l) + \sum_{n=l}^{N} \varphi\left(\frac{x + n - l}{N+1} - m\right) - \sum_{n=1}^{N} \varphi\left(\frac{n}{N+1}\right) \\
&= (m+1)l + \sum_{n=N+1-l}^{N} \varphi\left(\frac{x + n}{N+1} - m - 1\right) \\
&\quad + m(N-l+1) + \sum_{n=0}^{N-l} \varphi\left(\frac{x + n}{N+1} - m\right) - \sum_{n=1}^{N} \varphi\left(\frac{n}{N+1}\right) \\
&= \sum_{n=N-l+1}^{N} \left\{\left[\frac{x+n}{N+1}\right] + \varphi\left(\frac{x+n}{N+1} - \left[\frac{x+n}{N+1}\right]\right)\right\} \\
&\quad + \sum_{n=0}^{N-l} \left\{\left[\frac{x+n}{N+1}\right] + \varphi\left(\frac{x+n}{N+1} - \left[\frac{x+n}{N+1}\right]\right)\right\} - \sum_{n=1}^{N} \varphi\left(\frac{n}{N+1}\right) \\
&= \sum_{n=0}^{N} \phi\left(\frac{x+n}{N+1}\right) - \sum_{n=1}^{N} \phi\left(\frac{n}{N+1}\right).
\end{aligned}
$$

To prove that ϕ is increasing fix $x < y$. If $[x] = [y]$, then

$$\phi(x) = [x] + \varphi(x - [x]) = [y] + \varphi(x - [y]) \le [y] + \varphi(y - [y]) = \phi(y),$$

and if $[x] < [y]$, then

$$\phi(x) = [x] + \varphi(x - [x]) \le [y] \le [y] + \varphi(y - [y]) = \phi(y).$$

It is clear that ϕ is continuous at every point of the set $\mathbb{R}\backslash\mathbb{Z}$. If $k \in \mathbb{Z}$, then by the continuity of φ and (1) we obtain

$$\lim_{x\to k^+} \phi(x) = \lim_{x\to k^+} \big([x] + \varphi(x - [x])\big) = k + \lim_{y\to 0^+} \varphi(y) = k = \phi(k)$$

and

$$\lim_{x\to k^-} \phi(x) = \lim_{x\to k^-} \big([x] + \varphi(x - [x])\big) = k - 1 + \lim_{y\to 1^-} \varphi(y) = k,$$

which completes the proof. □

5. Matkowski–Wesołowski case

First of all observe that formula (14) with $N = 1$ coincides with formula (2). So the main part of assertion (ii) of Theorem 1.1 is a very special case of Theorem 3.6, whereas its moreover part follows from Corollary 4.2. Now we would like to get a little bit more information about the class $\mathcal{C}$. For this purpose, we denote the convex hull of a set A by $\operatorname{conv}(A)$ and put

$$\mathcal{W} = \{\varphi_p : p \in (0,1)\},$$

where $\varphi_p\colon [0,1] \to [0,1]$ is the function defined by (2).

Proposition 5.1. *The set $\mathcal{W}$ is linearly independent. Moreover:*

(i) $\operatorname{conv}(\mathcal{W}) \subset \mathcal{C}$;
(ii) $\operatorname{conv}(\mathcal{W}\backslash\{\varphi_{\frac{1}{2}}\}) \subset \mathcal{C}_s$.

Proof. To prove that $\mathcal{W}$ is linearly independent fix $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $0 < p_1 < p_2 < \cdots < p_n < 1$ and assume that

$$\sum_{i=1}^{n} \alpha_i \varphi_{p_i}(x) = 0,$$

for every $x \in [0,1]$. Applying (2) we conclude that $\varphi_{p_i}(\frac{1}{2^k}) = p_i^k$ for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$. Then for every $k \in \mathbb{N}$ we have

$$\sum_{i=1}^{n} \alpha_i \left(\frac{p_i}{p_n}\right)^k = 0.$$

Taking the limit as $k \to \infty$ we get $\alpha_n = 0$. Repeating this procedure $n-1$ times gives $\alpha_n = \alpha_{n-1} = \cdots = \alpha_1 = 0$.

Assertion (i) follows from Remark 2.1 and assertion (ii) is a consequence of the moreover part of assertion (ii) of Theorem 1.1. □

To formulate an answer to the problem posed in [9] by Janusz Matkowski define first a function $\varphi_1\colon [0,1] \to \mathbb{R}$ putting $\varphi_1(x) = 1$ and observe that by Proposition 5.1 and the fact that $\varphi_1(0) = 1$ and $\varphi_p(0) = 0$ for every $p \in (0,1)$ the set $\mathcal{W} \cup \{\varphi_1\}$ is linearly independent. Let $\mathcal{M}$ denote the vector space whose basis is $\mathcal{W} \cup \{\varphi_1\}$, i.e.

$$\mathcal{M} = \operatorname{lin}\big(\mathcal{W} \cup \{\varphi_1\}\big).$$

Applying Proposition 5.1 and Remark 1.2, we get the following result.

Theorem 5.2. *Every function belonging to $\mathcal{M}$ is a continuous solution of Eq. (e_1). Moreover, $\sum_{i=1}^{n} \alpha_i \varphi_{p_i} \in \mathcal{M}$ is:*

(i) *monotone provided that $\operatorname{sgn}(\alpha_i) = \operatorname{sgn}(\alpha_j)$ for all $i, j \in \{1, \ldots, n\}$ such that $p_i, p_j \in (0,1)$;*
(ii) *singular for all $p_1, \ldots, p_n \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$.*

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