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Mathematics and Philosophy*

My statement is supposed to be an encouragement, an introduction to a discussion about connections between philosophy and mathematics. Such an introduction should contain statements provoking a response. In this case it is not too difficult. I just look at mathematics slightly differently than most of my colleagues. That is the case perhaps because I came to mathematics "from outside." I was interested in biology and economics, I studied physics and only later did I become a mathematician.

I am of course aware of the fact that the relationship between philosophy, mathematics and reality is difficult and that I may be wrong in many, or maybe even in all my points. I would gladly correct my views upon hearing appropriate arguments. However, what I will say here is what I deeply believe in, not just a way for me to sate my natural need to formulate everything as strictly as possible.

I divided my statement into six theses. Of course the division is quite arbitrary and some of the points are closely related.

Thesis 1. For simplicity I will use the words of Hugo Steinhaus, one of the greatest mathematicians of our century, who in one of his books said that

- **The subject of mathematics is reality,**
- **Mathematics is universal.**

Of course, it is a wonderful summary, but just a summary. Speaking more precisely I believe that mathematics is just the structure of our world. Not a description of that structure, but the structure itself. Undoubtedly, a mathematician can create very strange objects and it may seem he is straying far away from reality. It only appears that way. If his mathematics is good, it will sooner or later turn out it is a fragment of reality. If it is bad, it will only be an amalgamation of bits of the real world, like a dream is an amalgamation of our everyday experiences. A dream may be strange, but have you noticed how you cannot speak a language you do not know?

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I have been asked whether mathematics would be different if the world was different. Of course, it would. Moreover, I think if there was no world, there would have been no mathematics either – in no sense of the world. One can conceive worlds without mathematics, but not ours. The worlds seen by mystics, felt by dreamers and poets may be nonmathematical. Our hard, real world is a world of mathematics. Mathematics in its broadest sense teaches us that certain things are impossible. If two sides of a triangle have lengths $a = 3$ cm and $b = 4$ cm, it is impossible for the third length c to be shorter than 1 cm or longer than 7 cm. If the two sides of lengths a and b are perpendicular, all the possibilities except $c = 5$ cm are excluded (Pythagoras theorem). That is why we do not observe miracles. They are just impossible.

Thesis 2. We are currently observing very fast, intensive development of mathematics. It is displayed in the exponential growth of the number of mathematical papers, bulletins and researchers. It is particularly spectacular how we have had more mathematicians in the twentieth century than in all the previous ones. It is known that after World War II we have proven millions (sic!) of new theorems.

This is an appearance, a camouflage. No-one can apply millions of theorems. These are naked king's robes. At the beginning of this century we axiomatized probability theory, created topology, functional analysis, several disciplines forming the foundations of modern mathematics. Nothing really new has been discovered since. We are playing with the same ingredients: set, relation, function, properties of functions. The speed of mathematical expansion of mathematical methods, which engulfed physics and chemistry in recent centuries and have been serving astronomy from the beginning, has been halted. Attempts to apply mathematics to study social phenomena are a compromitation and those to describe biological systems an undertaking beyond mathematicians' capabilities.

A similar phenomenon can be observed in entire science, technology and the whole of civilization, perhaps with the exception of biology. Let us notice that we have created some new ways of transport between the nineteenth and twentieth century, that is, a bicycle, a motorbike, a car, an airplane, a helicopter at the end of the 30s. We have been perfecting their construction, developing production, we are englufing earth in a network of motorways, making runways longer, but for over half a century nothing new has been created. Absolutely nothing.

Let us then come back to mathematics. Why is it spreading around instead of rising up? There are two reasons in my opinion, namely, a psychological and a sociological one. First of all, modern mathematics is beautiful, strict and flexible already, we can express many things in it. It is good. Good is the biggest enemy of better. We are not looking for anything new inventions, because it is easier and more enjoyable to develop old and beautiful theories which still bear fruit. Similarly, it is easier to perfect cars and build highways rather than think of, construct and apply a new way of transport.

Secondly, in the past a researcher was a bit of a madman, a bit of a freak and a bit of a hermit. Today, a researcher is a stable and calm profession. More stable and calm than the profession of an advocate, businessman or even a farmer. Hundreds of thousands of people are researchers. Powerful institutions have been created that make sure that a researcher's profession is only undertaken by people of appropriate qualifications. Methods of awarding scientific advancement and securing grants have been codified. Forms which need to be filled by every researcher are getting longer and more complicated and an ability to fill them is one of the fundamental skills to develop one's career and secure grants.

In this way, the strange ones and some of the cheating ones have been cut away, completely thrown out. But by throwing out thousands of non-conforming eccentrics, we have thrown out many geniuses with them. Therefore, the realization of groundbreaking ideas is now impossible.

This is not a joke or a paradox. One of my best friends, an American mathematician, applied for two grants. In the first one, about his side interests, he proposed a research project within a well-known and intensively developing approximation method. He supported that application with a literature review and obtained a grant without a problem. In his second application, connected to his life's passion – stochastic processes theory, he proposed to study a brand new, interesting property of Markov processes. He did not get the grant. What is even worse, the feedback following this rejection can be summarized in the following way: "You asked to study something new, that has not been studied by anyone before you. It would be better if you studied things already being researched by well known mathematicians" (and some names were given here). That friend gave me that letter to read and I thought that taxpayers' money was spent to build an institution which will successfully stop every groundbreaking result in science.

The scientific bureaucracy, while taking part of the money given by the society for the development of science, is interested only in one thing – creating nonsensical rules and decisions, which successfully contribute to wasting the rest of the funds, not secured by the officials.

Thesis 3. Progress can be slowed down, but not halted – especially not scientific progress. There will be people who will change modern science, especially mathematics. In several decades, we will be dealing with new notions and efficient methods which we cannot even dream of now. It is not that the quantity will translate into quality. Quite the opposite, new quality will arise despite the quantity.

That sort of mathematics is sought for while dreaming of universal methods of catastrophe theory, fuzzy sets, fractal geometry or chaos theory. So far we only have "ground trials," but there is hope for liftoff. They are also hugely controversial. Let us just recall the confusion surrounding catastrophe theory. Some saw in it a universal way of describing the dynamics of the world, which allows one to understand almost anything – from the formation of spirals on a slug's shell to the workings of the heart and prison strikes. Others just considered it to be an unremarkable fragment of singularity theory on manifolds.

Similar controversies, although on a smaller scale, surround the fuzzy set theory. Its opponents consider it to be complete nonsense or a badly founded fragment of probability theory. There are also those who would like to make the study of it illegal via appropriate law bills or decisions of scientific officials (!). Its supporters think that systematic usage of fuzzy set theory will give a technological advantage over the rest of the world to Japanese engineers.

In both cases the truth lies somewhere in the middle, but it seems like the sceptics are closer to it this time. I have great hopes in fractal theory, an optimist that I am. Thanks to this it was noticed that the hardest, most complicated processes seem to be describable as dynamical systems on spaces of measures. Both relatively simple chemical substances and complex biological systems seem to be fractals.

Thesis 4. Mathematical discoveries have an important philosophical meaning. This does not only concern discoveries in foundations of mathematics. Modern mathematical analysis, dynamical system theory and probability theory touch upon such fundamental issues as the determinism problem, the possibility of exact prediction of events, the existence of hidden parameters, etc.

Thanks to the work of Edward N. Lorenz and the general dynamics of stochastic processes the illusion of the predicted, deterministic world has been shattered. In short, Lorenz's reasoning goes as follows. Imagine a system described just by one parameter. Let us call it x. What is more, we assume that the values of x are bounded, being, say, in the interval $[0, 1]$. Moreover, let us assume that x changes in every time unit in a completely deterministic way. In mathematical language, this means that our model is described by a recurrence equation:

$$x_{n+1} = T(x_n), \quad n = 0, 1, 2, \ldots \tag{1}$$

where x_n denotes the value of parameter x at time n and T is the given transformation of interval $[0, 1]$ into itself. Thus, we conceived a reasonably simple space: the $[0, 1]$ interval. Let us also take a relatively simple function T. If we discard linear transformations, which in the one dimensional case yield completely uninteresting dynamics, the simplest map is a second degree polynomial. One such polynomial is, for example, $T(x) = \lambda x(1-x)$, where λ is a fixed parameter. For $0 \leq \lambda \leq 4$ this maps the $[0, 1]$ interval into itself. Thus, we will study the following recurrence equation:

$$x_{n+1} = \lambda x_n(1 - x_n), \quad n = 0, 1, 2, \ldots \tag{2}$$

for values $x_n \in [0, 1]$ with a parameter $\lambda \in [0, 4]$. It turns out that already such as simple system has multiple unexpected properties. It can imitate almost all types of phenomena occuring in nature. One can obtain sequences (x_n) which increase or oscillate and converge to one point. One can obtain periodical behavior with any desired period, for example 17 in the case of a 17-year cicada. Furthermore, bounded, aperiodic and irregular trajectories are also possible. We call them chaotic trajectories. It is simplest to obtain them by taking $\lambda = 4$. Then for almost all $x_0 \in [0, 1]$ (in the Lebesgue measure sense) the trajectories will be chaotic. They have a certain particular property. If we input the starting value x_0 with, let us say, 20 decimal places accuracy and we start calculating x_n from the formula (2) with $\lambda = 4$, then x_{90} may contain an error already on its first decimal place. With each step the initial error will approximately double. We will run into the same issue if we input x_0 exactly, but calculate x_n on two different computers with different rounding algorithms. The result of calculating x_{90} may be meaningfully different. This exceptional sensitivity to change of initial conditions and, in particular, to rounding errors during calculations is characteristic to chaotic processes. It makes it impossible to predict them on a longer time period.

The situation would be completely different if we repeated those numerical experiments for $\lambda = 2$. In each case we would have obtained $x_{90} = 0.5$ with high accuracy. That is because for $\lambda = 2$ the dynamical system given by (2) is asymptotically stable.

We have noticed, following Lorenz, that chaotic and stable processes appear in a very simple, one dimensional system. Those types of processes also feature in multidimensional processes describing complicated models from nature. Movements of large masses of water and air unfortunately belong to the class of chaotic processes. This is why forecasting weather, say, a month ahead may turn out to be impossible even if we learn all the laws governing our atmosphere. Similarly, changes of parameters describing the state of an organism undergoing some diseases belong to the class of chaotic processes. Thus even when knowing perfectly the behavior of the illness, a patient's fate may be uncertain. In short, we can say that knowing the laws governing the world does not guarantee its predictability.

Let us look at those issues in a slightly more general way. Probabilistic theory teaches us that every deterministic process can be approximated to arbitrary precision by random processes. This is in essence a corollary to the large of law numbers and central limit theorem. A book on a table seems to be motionless. In reality it is being bounced on an enormous amount of vibrating atoms of the table. Those hits, on a macroscopic level, give the effect of a hard, unyielding surface.

On the other hand, it has been shown that a wide class of stochastic processes is a limit of purely deterministic processes. Let us again consider (2) for $\lambda = 4$. Let us pick an arbitrary real number $x_0 \in [0, 1]$, compute the successive values x_n and define a sequence ξ_n by

$$\xi_n = \begin{cases} 0 & \text{for } x_n \le \frac{1}{2} \\ 1 & \text{for } x_n > \frac{1}{2}. \end{cases}$$

Such a sequence ξ_n is indistinguishable from a sequence obtained by successive random trials, for example coin tosses ($1 =$ heads, $0 =$ tails). No statistician would be able to tell those sequences apart, and a pure mathematician could prove that for almost all $x_0 \in [0, 1]$ such a distinction is not even possible. More generally, it has been shown (C. W. Kim, A. Iwanik) that every Markov operator can be approximated in strong topology by operators coming from deterministic transformations.

Therefore, whether our observations and deliberations drive us to a conclusion that the world is governed by deterministic or random laws, it may be an illusion brought by the finite precision of our instruments.

To finish it off, one more remark from classical dynamical systems theory. Let us recall that a classical dynamical system (with continuous time) acting on space X is a family $\{S_t\}$ of transformations of X into itself, indexed with a real parameter $t \in \mathbb{R}$, which satisfies the following:

$$S_0(x) = x \quad \text{for } x \in X, \tag{3}$$

$$S_{t_1+t_2}(x) = S_{t_2}(S_{t_1}(x)) \quad \text{for } x \in X, t_1, t_2 \in \mathbb{R}. \tag{4}$$

This system is of course interpreted as movement in the space X. Every point is moved in such a way that its position at time t is given by $S_t(x)$. For a given x, the function $t \mapsto S_t(x)$ is called the trajectory of x. It is denoted in short by $(S_t(x))$. Condition (3) tells us that at $t = 0$ the trajectory comes through point x and (4) that points in X move between time t_1 and $t_1 + t_2$ in the same manner as between $t = 0$ and t_2. Thus, a dynamical system may be interpreted independently from the laws of time on X.

It is easy to see from (3) and (4) that not all curves in X can be trajectories. They can only be functions which are constant ($S_t(x) = x$ for $t \in \mathbb{R}$), periodic ($S_{t+T} = S_t$ for $t \in \mathbb{R}$) or injective ($S_{t_1}(x) \neq S_{t_2}(x)$ for $t_1 \neq t_2$). Let Y be an arbitrary set. The trace of a trajectory $(S_t(x))$ on the set Y is any function $f\colon \mathbb{R} \to Y$ of the form

$$f(t) = \phi(S_t(x)) \quad \text{for } t \in \mathbb{R},$$

where ϕ is a certain mapping from X to Y. We call ϕ the projection of X onto Y. We can now ask a natural question: which functions $f\colon \mathbb{R} \to Y$ can be traces of some dynamical system? The answer is brief: all of them. More precisely, for every family of functions $f_\lambda\colon \mathbb{R} \to Y$ indexed by a parameter $\lambda\,(\lambda \in \Lambda)$ there exists a space X and a dynamical system $\{S_t\}$ on X such that all functions f_λ are traces of trajectories of this system with the same projection ϕ independent of λ.

The philosophical implication of this theorem is clear. Whatever strange, undeterministic or plainly incomprehensible phenomena we observe in our world Y, there may exist another, completely deterministic world X and what we observe is just a projection of the dynamics of a different world.

Thesis 5. The most important riddle in the world – the essence of consciousness – is not a mathematical problem. Mathematics belongs to the sphere of matter, not the sphere of the soul. Reasoning and proving is an act of the brain, which – like a computer – is built from material elements, neurons in this case. It is useful to recall here that the engineers of a new generation of computers are trying to mimic neural networks. "WE know mind and body," says Nisargadatta Maharaj. They are embedded in the material world in which mathematics is the king. Our consciousness is outside of the material world, which is why I believe it is unattainable for mathematics.

What is consciousness, I know not. But the certainty of my existence, my separation from other people is so strong that no reasoning and no observation can damage it. Let us assume someone proves to me that my consciousness does not exist, that I am merely a collection or a construction of atoms, or that there is no me at all. I will look at him with admiration, or compassion, depending on the standard of his proof, but it will be ME looking.

It is strange that only a small number of people feel the riddle of their own existence. For those who do not feel it, words of Steinhaus, who writes about his consciousness, may be useful again:

> When I accept the multitude of similar beings, then the astonishing fact that one of them, for some reason, is "me" and it's only that one that is hurt by pain or can enjoy a drink, becomes a vast riddle.

Thesis 6. Mathematicians mostly dislike and do not understand probability. In the survey of most important achievements of modern mathematics created by the Bourbaki group, there is no mention of probability theory. Meantime it is this theory that has had the most spectacular achievements. As far as theory is concerned, it is one of the most advanced parts of mathematics. As regards applications, it massively outclasses all the others.

The universality of probabilistic methods is astonishing. The applications of probability in physics, genetics, demographics, etc. is obvious to everyone. The applications in "pure" mathematics are less known. However, probability has become useful in number theory, differential and integral equations, numerical methods and many other parts of analysis. Thus, the solutions of some important differential equations are expressible as explicit mean values of some stochastic processes (Feynman–Kac formula). Other differential equations can be approximated by difference equations, which are most easily solved by random methods.

As a curiosity I will mention that the theory of the integral of multivalued functions has been developed by probabilists in a much more general form many years before it was rediscovered by analysts.

It is becoming less of a joke to say that probability theory is one of the main branches of nature science, and the rest of mathematics is a supporting science.

A question remains whether the development of probability theory is a reflection of the fact that our world, not even including quantum mechanics, is by its nature governed by random laws. I do not know, but I hope that is not entirely the case.

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