FORMAL LANGUAGES AND MODEL THEORETIC PERSPECTIVES IN PHYSICS*

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We show that recent big growth of applications of Category Theory to Physics might be associated with unavoidable appearance of model theoretical structures coming from formal languages used to describe mathematical models of so called physical reality. Even in the simplest case of Elementary Protocolar Theory we are (to fulfil the conditions of consistency and simplicity of the language) confined to some model theoretical limitations. We discuss some examples. We also formulate conjectures and perspectives for future investigations.

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1. Does physics need abstract tools of Model Theory or Category Theory?

One, quite obvious answer is: No. As any other abstract mathematical tool, this might be avoided just by simple keeping track of experiments and their direct data. Let us try to describe what it means that we have theory based on direct experimental data. First, we choose language as simple as possible (but consistent) whose individual variables refer to (finite) numbers of experiments. So, our sentences corresponding to experiments are to be formulated in the language of the so called first order predicate logic and consistency of the sentences is defined with respect to this logic. The choice of first order logic is motivated by minimal theoretical entanglement of this logic (see Section 2) and ability to speak directly about set of experiments and its results. We have to speak about some logic because of the consistency

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of theory we wish to develop. Henkin had proven the completeness of the first order predicate logic by means of explicit construction of some models for theories in this language [34]. In particular it was proven:

Theorem 1 *If T is a consistent theory in the language of first order predicate logic and if m is a cardinality of the set of its primitive symbols, then T is satisfied in some domain of cardinality m.*

In particular for $m = \aleph_\circ$ we have original Löwenheim–Skolem Theorem [32]. In fact it holds: [36]

Theorem 2 (Lindström) *First order logic is the only one which is closed with respect to $\wedge, \neg, \exists$ and such that theorems of Compactness and Löweheim–Skolem hold.*

Next let us suppose we are performing n experiments and as a result we have a set of values for set of observables to be measured in each of them. We want to describe the language which would enable one to speak in consistent and direct way about this situation and which would be as simple as possible. Thus the set of symbols of the language should contain

set of symbols (variables) for measurements: $m_1,\ m_2, \ldots$

set of symbols (variables) for the results of measurements (numbers): x_1^m, x_2^m ...

set of symbols (variables) for the observables to be measured: ob_1, ob_2, ...

set of predicate symbols for expressing that in measurement m_i we have measured an observable ob_j and we are obtaining x: $x(ob_j,\ m_i) \ldots$

We require that the (intended) interpretation of the theory in such language is the structure where domains

for m_i are measurements n_i, $i = 1, 2, \ldots$

for x_j are numbers r_j, $j = 1, 2, \ldots$

for ob_k are observables Ob_k, $k = 1, 2, \ldots$ for any natural number of performed measurements.

The theory in the above language enables one to speak about performed measurements and to catalogue their results. This theory is to be called Elementary Protocolar Theory (EPT). Its intended models are simply measurements — any finite number of them along with their results, corresponding to some finite number of observables. None of the finite number of experiments is to be expressed by the axioms of the theory so, this theory should have models in any finite cardinality. But being the first order it holds: [9]

Theorem 3 *If theory T in the first order language has any arbitrarily large finite model it has also infinite model.*

So, our theory speaks as well about infinite number of measurements and about collecting their results. But in fact we have:

Theorem 4 *Any Elementary Protocolar Theory has nonisomorphic nonstandard models.*

Proof: this is just reformulation of Löwenheim–Skolem theorem and observing that if M, N are two infinite models of the Theory of different cardinalities then M is not isomorphic to N.

The peculiarity of this Theorem comes from our intention to formulate EPT which would be expressing uniquely the simple situation of cataloguing experimental data (any finite) in a consistent way; this is not possible and in fact we are talking simultanously about any number of nonisomorphic domains where our theory is fulfilled. This is unavoidable.

We do not list up explicitly nonlogical axioms of the theory; we simply assume they do exist and they would enable us to fulfil our minimal requirements — to talk consistently about any finite number of experiments.

If such a theory does not exist it means we cannot talk about any finite number of experiments in a direct (first order logic) and formally consistent way. This is even worse than to have nonisomorphic models: the ability of constructing a formally consistent physical theory based on experiments is questionable.

Now let us suppose we are performing some number of experiments, say $k \in \mathbf{N}$. In k-th measurement we are getting a natural number k as an output. We are formulating some sentence (predicate) in the first order language about natural numbers. It holds:

Theorem 5 *Suppose, that we have countably infinite number of experimental outputs, which are interpreted by Standard Natural Numbers. With each output we associate some predicate related to natural numbers (if only it is expressible in the first order predicate language, where each Standard Natural Number has a name). Also, with each output we associate some true formula of Peano Arithmetic. Even then, there is no consistent Theory (in the language of the first order logic), which would be generated by the above sequence, and which would allow one to describe what Standard Natural Numbers are.*

Proof: the theory ($\mathbf{N}_0$,S, $+$, $\cdot$) of the structure of Standard Natural Numbers is not axiomatizable.
The explanation of this is due to essential incompleteness of Peano Arithmetic (PA) which is axiomatic, first order theory of natural numbers. In fact it holds: [11]

Theorem 6 *PA has 2^{ω_o} nonelementary equivalent countable models.*

This phenomenon is a direct consequence of Gödel Incompleteness Theorem [37]. From the other side PA has unique Standard Model which is the structure ($\mathbf{N}_o$,S, $+$, $\cdot$) where $\mathbf{N}_o$ is a set of all natural numbers with successor functional symbol S. The theory Th($\mathbf{N}_o$) of all true first order sentences in the standard structure can be characterized: [8]

Theorem 7 *The Theory Th($\mathbf{N}_o$) is hereditarily undecidable.*

Notice, that hereditary undecidability is stronger condition than undecidability [8]. We also know that: [11]

Theorem 8 *If Theory T is consistent, complete and axiomatizable then T is decidable.*

Th($\mathbf{N}_o$) is complete because all its true sentences are true in the model $\mathbf{N}_o$ (from definition); it is also consistent (it has a model $\mathbf{N}_o$). Then we can conclude:

Theorem 9 *Th($\mathbf{N}_o$) is not axiomatizable.*

So, although we have Standard Model for PA which is unique we cannot use any recoursive set of sentences (in first order language) generating the theory Th($\mathbf{N}_o$). PA, being described formally by a recoursively axiomatizable means, allows infinitely (continuous) many nonisomorphic countable models. In fact we have proved:

Theorem 10 *One cannot give an infinite, in fact, recoursively enumerable list of sentences in the language of the first order predicate logic describing experimental outputs which would be expressing the theory of all natural numbers which order that list.*

The order and recoursive character of infinite experimental data cannot be explained by the theory generated by protocolar sentences associated to the data, even in the case when every successive experiment is producing successive natural number. Even infinite sequence of experimental protocolar sentences plus PA are not able to cover formally all true sentences in the structure ($\mathbf{M}_o$, S, $+$, $\cdot$), where $\mathbf{M}_o$ are natural numbers indexing experiments under consideration.

Theorem 11 *We are performing infinite countably many experiments generating predicates $\{P^n_\theta(x_n)\}_{n=1}^{\infty}$ about natural numbers. We cannot generate the theory of standard natural numbers (even for every n we have $x_n = n$).*

This theorem announces an important thing: we cannot treat experimental data literally. Sentences $\{P_\theta^n\}(n)$ expressing that n is just n, for every n, do not express that we are in fact dealing with unique domain of standard natural numbers. So, literal and consistent treating of experimental data is not fully legitimate. This is also a hint toward necessity to use Model Theoretic analysis, at least in some cases when we are to formulate formal theories with intention to be very tightly connected to experiments.

Even if we are treating every measurement and every result corresponding to it (expressed by predicates) as axioms we are formally talking about noncountably many nonequivalent domains and there is no way to improve it by performing more experiments and adjoining more protocolar sentences as axioms (in fact any first order) into the language of the theory.

We can, of course, avoid this strange behaviour just by considering only a finite number of formal sentences, say k, corresponding to finite number of experiments but again we can try to built the formal, consistent theory (in first order language), which would be about those protocolar sentences (about any finite number of them), having arbitrarily large finite intended models; the reason is the statement: 'We have exactly k-finite events' is not a logical axiom and is not nonlogical as well (we do not formulate the theory just about k events and not about $k+1$, there is no logical reason for that).

Also, assuming that we already know well what the set of all natural numbers is (or any other mathematical object is) not caring about their not unique formal description as a first order theory, we are obtaining the so called many-sorted logic [36]. This logic can be equivalently formulated as a first order [36] and Löwenheim–Skolem theorem also holds for it (althought compactness theorem does not hold).

If it is about Set Theory it can have axiomatic form, so called Zermelo–Fraenkel Set Theory (ZF) possibly with addition of the axiom of choice (ZFC). ZFC is the theory in the first order language without any functional or constant symbols; the only predicate symbol is binary $\in$-symbol, expressing property of being an element [10]. As usual we have countably many individual variable symbols. As the first order theory (possibly consistent) it posseses also a countable model (from Löwenheim–Skolem theorem). This is a little bit strange but let us notice that this model can be standard in the sense of accordance of its internal E-symbol (expressing being an element) to external, usual $\in$-symbol [38]. Moreover, this model can be standard with respect to equality which can be the same as external one (normal model). Then we see that for ZFC there does not even exist a unique Standard Model and the variety of its standard and nonstandard models is realy huge [10, 33]. ZFC does not respect our intention to speak uniquelly about some standard universe of sets. Moreover, we can define Natural Numbers in ZFC (*e.g.* [46]). If one tries to regard ZFC as a basic theory this would lead to relativisation phenomena even for finite natural numbers.

2. Higher order theories

We have seen that in some sense first order theory of Standard Natural Numbers is not experimental — it cannot be generated by a countable number of first order axioms coming from the experiments.

Turning to the higher order theory one becomes immediately equipped with some additional theoretical tool. The second order theory is the one based on the language allowing, roughly speaking, for quantifications over subsets of some set M and over functions $F : M \times M \to M$. The first order theory allows for quantifications only over elements of set M.

Some notions which are not unique in first order theory in higher order theories sometimes become unique (up to isomorphism). A good example is the ordered field of real numbers *i.e.* the structure $\mathbf{R} = (\mathbf{R}, +, \cdot, <, 0, 1)$. We have: [36]

Theorem 12 *There does not exist a set of axioms in the first order logic which would characterize* $\mathbf{R}$ *uniquelly up to isomorphism.*

Proof: it is direct consequence of Löwenheim–Skolem theorem and observation that language of the first order logic is countable and the number of first order formulas true in $\mathbf{R}$ is countable. Then there exists a countable model; this model is not isomorphic to $\mathbf{R}$.

But it is known that second order logic characterizes $\mathbf{R}$ uniquely up to isomorphism [36].

The axiom which cannot be formulated in first order logic is the completeness axiom:

$$\forall X \subseteq \mathbf{R}(if \neg(X = \emptyset) \text{ is bounded, then } X \text{ has least upper bound})$$

This axiom is dealing with universum of Set Theory by taking use of power set operation. This phenomenon is characteristic for higher order theories — we have to rely on the set theory which in turn has very nontrivial spectrum of its models. So, we can sometimes avoid nonuniqueness of description of some mathematical structures (in first order language) by appealling to higher order language but at the expense of dealing with nonunique models coming from set theory. (By the second Gödel Incompleteness Theorem [37] the statement 'ZFC has a model' cannot be proved by formal means in ZFC itself, but if there does not exist any model of ZFC at all, ZFC would be inconsistent [10].) For higher order logics we have also many nonisomorphic nonstandard models which were originally constructed by Henkin [35]. He also showed that the higher order logic is not complete without taking into account these nonstandard models. Higher order logic is complete only with respect to its nonstandard models. Hence, we see that we are expressing some uniqueness of mathematical structures in higher order theories by use of the language which is not complete by itself with respect to standard structures [45].

There exists so called Hilbert thesis [36] which states that it should be possible to translate all mathematical statements (nonlogical) into first order language and proofs (provability relations), even nonformal, which exist in mathematics, would become formal in the sense of first order logic.

This is also plausible that most of the formal, based on mathematics, reasonings in physics could be (in principle) expressed in first order logic. Every day language used in physics does not care about its order and it is rather arbitrary mixture of orders and self referential expressions. But this does not change the fact that in the end physics tries to refer to formal reasonings which are dependent on distinguishing orders and as a result to situations sensitive for theory of models.

3. Why do physicists in practice avoid model theoretic reasoning?

We list here some reasons answering the question in the title.

1. Construction of models for the first order theory (and not only [49]) might be performed by the use of closed well formed formulas (cwff) of the language of the theory; there is an infinite number of them. But in practice we are conducting physical analysis in finitistic way, using only finite number of sentences from the every day language. Even formal languages are considered finitistically. From the other side we freely analyse infinite numbers of various formal objects like quantum states, dimensions of Hilbert space, particles, degrees of freedom *etc.* So, the finiteness is only with respect to formal languages used.

2. We do not analyze the situation of having infinite number of formulas or sentences which are expressing something in nonintended way, which seems to be unavoidable because of the formal language used.

3. We do not explore the structure of the sentences; they apperently look like being transparent. Because of this the logic also seems to be transparent.

4. We treat mathematics as a closed, ready to use, given for granted system of procedures which can or cannot be used for specific applications.

5. The above facts always allow one to be in a distinguished position; from one side we are outside of the internal problems and techniques specific to mathematics. We use only procedures generated somewhere

outside and we have direct access to them by using transparent language. From the other side we have always direct access to experiments and their results which is possible also because of the transparency and directness of the language.

6. Because of the assumed transparency of the language (formal) what we freely use is not a formal language but the one which mixes orders, logics and is selfreferential.

7. Also because of the transparency of the language we treat objects and its names as equally managable.

Those conditions are incidentally characteristic also for classical descriptions of the physical world. The classical language seems to be nonappropriate to describe some Quantum Mechanical phenomena. That is why, the main field where Model Theory would find applications is Quantum Mechanics.

4. Some arguments for using tools of Model Theory in physics

Although we have given some evidence for nonuniqueness of formal descriptions of some simple situations dealing with making experiments and cataloguing their results, we still do not see the necessity of using tools of Model Theory in physics. Here we collect arguments for the existence of appropriate place for them in physics.

First: physics is not free of difficulties; some of them are very basic (*e.g.* Quantum Gravity and its background independent formulation [48]).

Second: the arguments coming from Model Theory are purely mathematical in fact, and, in principle, could be used in physics (and in fact they are [see Section 5]).

Third: every day physical practice is to built various mathematical models of the so called reality. Language is also a part of the reality (with its well established formal shape). Why we do not need at all the models of the language in a correct mathematical modelling of the reality?

From the discussion of the previous paragraphs we can formulate the general rule of transparency of the language:

General Rule 0 *We are talking directly about the results of experiments and we are building direct mathematical models of the reality. Precision of the measurement is the only obstruction for models to be perfect.*

We clearly see that the above rule is not valid in QM. There are inherent reasons for measurements not to be precise, it does not matter how precise measuring devices are. It might be that this is connected with the opposite rule of nontransparency of the language and this would show exactly the place where Model Theory might be applied? We can formulate the possible rule as follows:

General Rule 1 *Formal languages (and also physical reasoning based on them) are talking directly about their models, not about reality, usually are not uniquely determined. Experiments and their results are given to us by nonavoidable protocolar formal language. This language is associated to them as a formal language to its models.*

Let us observe that the language of QM is not usually directly connected to the so called reality: we are talking about quantum states, or amplitudes for example which are not observable in principle. They have only theoretical meaning as a nondispensable part of the structure of the theory. Observables, even commuting lose their meaning as having assigned values before experiment (see Kochen–Specker theorem in Section 5). One cannot measure non commuting observables simultaneously with the arbitrarily high precision. All this is not just failure of our description and/or lack of precision of the measurements but they are very features of reality — as if formal language would become the part of it. One proposition for realization of these ideas in context of QM can be found in Section 5. It is also well known that logic generated by QM is not classical [51]. The so called Quantum Logics are intensively explored, and also from this perspective [7] the connection between QM and Category Theory is evident.

Anoher clue for using tools of model theory comes from existence of highly theoretical (speculative) branches of physics. There are: Superstring theory (so called M-theory and related AdS/CFT correspondece, the principle of Holography), Quantum Gravity approaches (background independent theories [loop QG and categorification], causal sets) or even Cosmology.

Formal mathematical models require formal languages which are subjects to Model Theory. Those branches of physics are not so tightly connected with experiments and formal aspects like internal consistency of the theory plays a big role here.This is the reason why modelling of the formal language might be also a valuable tool (see *e.g.* [43, 44]). Let us notice an important feature of a model theoretical approach: the formal language becomes an object which in turn is investigated. From mathematical point of view we know that the object in question is special category which is called topos [49]. That is why the appearances of topos structure in physical theories (in essential way) give a strong hint toward ability of model theoretical analysis. Also, our dealing with various non trivial (which are not Set) Categories, in the context of physical considerations justifies Model Theoretical approach. Category Theory [7] is a correct language to talk about toposes which in turn allows model theory considerations [49].

We shall return to this point in Section 6. In the next section we give physical examples where Model Theory and Categories are essentially involved.

5. Examples

5.1. QM and toposes

It has recently been proposed [1,2] how the structure of the toposes arises in QM. It was achieved by restating the so called Kochen–Specker theorem in QM [3] in terms of nonexistence of global elements in a special topos $\mathbf{Set}^{\mathbf{W}^\star}$ (of sheaves of dual Boole'an subalgebras of the lattice of projection operators in Hilbert space H, dim(H)>2).

Theorem 13 (Kochen–Specker 1965, 1967) *Let Θ be a family of observables (self adjoint, linear operators) over Hilbert space H, dim(H)>2 such that identity $Id \in \Theta$ and let us consider functions (partial homomorphisms):*

$h:\ \Theta \to \mathbf{R}$ *such that*

whenever $A, B \in \Theta$ *and* $[A, B] = 0$ *and*

$h(AB) = h(A)h(B),\ h(\lambda A + \mu B) = \lambda h(A) + \mu h(B),\ h(Id) = 1$

then there does not exist global homomorphism

$h:\ \Theta \to \mathbf{R}$

compatible with partial ones.

Hence, no global assignment of real values to observables is possible in dim(H)>2. The Isham construction shows that nonexistence of compatible extension for partial valuations over all observables from Θ is expressible exactly as, so called functional condition for nonexistence of global element in the topos $\mathbf{Set}^{\mathbf{W}^\star}$:

$$\gamma_{|W_1}(\theta) = \gamma_{|W_2}(\theta)\,,$$

where γ is a global element and θ is any operator from Θ and from the common part of W_1 and W_2 as any subalgebras of Θ. This result is a direct indication for a deep relation between toposes — which are models of higher order (intuitionistic) logic — and QM. Also in the context of Quantum Gravity, Isham and Butterfield [1] have pointed out connections with toposes.

5.2. Models of ZFC and QM

We propose here a direct application of methods of Model Theory to QM which could be, at least, applied to interpretational investigations of QM. The work is in progress. Let us withhold the statement that Zermelo–Fraenkel axiomatic Set Theory with the Axiom of Choice- ZFC -speaks directly about "reality". It tells the truths about its models. Let us suppose that ZFC is consistent, so, some model of ZFC does exist. We know from Löwenheim–Skolem theorem and from collapsing Mostowski's lemma [10] that there exists Countable Transitive Model (standard) (CTM) — M.

The sentence φ: "M is countable" is not provable in M (if it were it would cause that every set in M was countable, but $2^{\aleph_o}$ is not countable in ZFC and also in M). It means that the function f: $\mathbf{N} \rightarrow M$('1 to 1') is not a set in M. Of course, from the outside any $\{x \in M|\Psi\}$ is countable as being a subset of $\mathbf{N}$.

We claim that discreteness of measured spectra of some physical observables might be connected to countability of Models of ZFC where also reals $\mathbf{R}$ are countable from the 'outside' (with respect to metatheory). Notice, that all sentences expressing any first order property of ZFC are valid as well in any countable model M. The set of all real numbers in M is also countable from the outside. So, measuring any real valued quantity in M gives us countable spectra. This observation is basic in trials to explain Quantum Mechanical phenomena via Model Theory. This is in total agreement with our General Rule 1 from Section 4.

In a direct way we can (by the use of formal theory) speak about models of the theory (not about the so called reality); that is why there are situations where our statements about what we actually measure are 'filtered' through the models of the theory. In the case of ZFC where almost all classical mathematics can be expressed [36] we have a very fundamental phenomenon comparable to generating Everettian worlds. Any output of the measurement is primarily connected (if it is about its set theoretical properties) to some model of ZFC (not necessarily to Standard one which is not formally distinguished but only intended).

To be more precise we need the method of so called forcing which was originally invented by Cohen [38] as a way to prove independence of Axiom of Choice and Continuum Hypothesis from the axioms of ZF.

From our perspective forcing is a passage from one model M of ZF(C) to another model N just by adding some set (or sets) not originally included in M. Such a set is called generic. If the Generic Set is just a subset of the set of natural numbers we call it generic real and this is what Cohen forcing adds to M (we call it also Cohen real). For countable models of ZF(C) the generic set Q always exists [10]. The procedure of forcing is a beautiful and nontrivial subject in itself and since 1963 it has been developed in various directions:

A. Boole'an Valued Models [29]

B. Model theoretic Forcing (finite and infinite) (Robinson in seventies [36]).

C. Categorical (topos theoretic Forcing) [30].

D. Forcing in so called Descriptive Set Theory [47].

In the context of our approach to QM we formulate the following rule showing the role of forcing;

General Rule 2 *Every output of the experiment (which is a real number) is given by a forcing which adds this real (reals) into some model (models) of ZF(C).*

This supposition does not mean we have a good knowledge of Models of ZF(C) under consideration. By the above supposition we can treat real numbers as being bounded by the forcing procedure, so, they are model dependent rather then reals which stay the same no matter what is the context in which they appear. All this requires more detailed analysis. The work on this approach is carried out by the author.

5.3. Non-Standard Analysis and Physics

Deep connections of QM and Model Theory were exhibited in the context of the so called Non-Standard Analysis (NA) [40]. NA was created and developed by Abraham Robinson in sixties [18]. Robinson had made a huge contribution to many branches of Model Theory [36]. NA is a direct application of nonstandard models (in the sense of first order predicate logic) of the theory of real numbers into mathematical analysis. This enables one, for the first time, to speak consistently about infinitely big and small quantities. The work of Farrukh shows usefulness and naturality of nonstandard notions (for example Nonstandard Hilbert Space) in the context of QM and its use of so called rigged Hilbert spaces and notorious use of δ-functions in QM. Much work on NA and QM was also done by Kobayashi [15, 16]. He had approached the problem of measurement by means of model theoretical constructions. Robinson also wrote some papers on physics and NA [17]. NA is about reals, so it obviously can be formally adopted into many branches of physics and mathematics, but usually (in the context of physics or analysis) it gives equivalent description and sometimes simplifies formal proofs [18]. The appearance of NA and its applications to physics are big clue toward correctness of the use of Model Theoretic methods in physics.

5.4. GR and toposes

There exists categorical approach to the analysis of infinitesimals so called Synthetic Differential Geometry [5]. The work of Moerdijk and Reyes [6] deals with models of infinitesimal analysis in toposes which are naturally generated in this context. The attempts to apply these ideas to physics were made in several papers [12–14]. They have tried to place General Relativity in the context of intuitionistic logic showing that the language of Synthetic Differential Geometry and toposes enables us to see various space-time solutions of the Einstein's Equations as just single varying object (in the sense of Lawvere [27]). Isham [1] also suggests the usefulness of Synthetic Differential Geometry in some approach to QG — so called Consistent Histories formulation.

5.5. The program of categorification

Explicitly, the program of categorification has been proposed by Baez [28]. In the eighties Abbhay Ashtekar [50] have introduced his new coordinates into GR. Since then it was possible to develop background independent nonperturbative quantum theory of gravity via so called loop QG [48]. This theory for the first time was able to produce explicitly solutions of quantum Wheeler–de Witt equations. Witten then conjectured [28] close connection of this solution to Jones invariant of links.

Many authors have produced invariants of 3-dim and 4-dim compact manifolds [21–23, 25]. It was along growing evidence for necessity to use abstract categories in this context (for example Hopf or Braided Monoidal categories). These invariants are widely used in connection with modelling of quantities (as transition amplitudes) of QG: from the one side we have triangulations of manifolds and colouring technique associating to it representations of groups and vertex operators resulting in calculations of invariants as traces in the so called tensor categories, but from the other side the invariant is a real topological invariant (it does not depend on triangulation used). For some reasons representations of groups should be replaced by Categories (Category of Representations of Hopf algebra) entering the higher order categories [23].

Very accurate language for considering QG in this context is the one coming from the so called Topological Quantum Field Theories where general cobordisms of the manifolds correspond to state transitions in QG. It was firstly formulated in axiomatic way by Atiyah in 1988 and developed in [28].

Categorification as an indispensable technique in QG was conjectured from the very beginning and then developed in a very promising direction by Crane [19, 20].

The necessity to use categories in this kind of considerations relies also on the peculiar fact (which is deeply Model Theoretic) that some consistent theories do not possess models in Set Category [42].

6. Conclusions and perspectives

6.1. First order categorical logic

The idea that language can be a kind of object which should be equally taken under consideration as any other object is not new but in the context of formal languages was realized not very long ago. It was originally done by Lawvere [26] who introduced categorical formulation of algebraic theories. Incidently Lawvere was one of the creators of topos theory and its wide applications [27]. To treat the language as any other object, seems to be strange (openness of the every day language) but on formal level this is crucial — we do not have simply mathematical structures described in a transparent language; the structure is effected by the language used. Clearly the equivalence of the so called coherent theory T (which is somewhat restricted version of the first order theory) with special category R_{T} (built of some formulas of the theory T) was done in [31]. So, to speak about models of T in a category (topos) Q we can equivalently speak about some functors M: $R_{\mathrm{T}} \to \mathrm{Q}$. Such a picture enables one to replace logic by categories. Also for special first order theories we can associate naturally toposes which are 'to classify' theories (Classifying toposes); they fully recognise the category of models (in toposes) of the theory. This approach enables one to develop investigation of models of theories and theories itself as the same kind of objects in unified way [4].

6.2. Higher order categorical logic

We have seen close connections of some physical theories with toposes which are special categories. From the perspective of the Model Theory, the toposes arose as natural objects whose internal language is the higher order intuitionistic one [49]. Henkin [35] proved completeness of higher order logic with respect to so called Nonstandard Models (the theory has enough nonstandard models to ensure that its theorems are semantically valid). Later it was realized that correct description of this phenomenon is just by models in toposes; moreover any (higher order) language generates topos T(L) whose internal language $\mathbf{L}(T(L))$ naturally interprets the language L [49]. So, the objectivisation of the languages (first or higher order) by means of Model Theory gives us toposes as objects to be considered in this context. That is why toposes which arose in the context of physical theories are hints for deep internal entanglement of Model Theory tools with some physical theories.

There is a big difference between intended domain which we want the formal theory to describe and true domain it deals with and at least in some cases it has to be taken into account also by physicists. Especially it could be done (in principle) in the context of:

- Anti-deSitter — Conformal Field Theory correspondence,
- M-theory, Dualities in Superstrings, Holography,
- Interpretations of QM,
- QG.

Detailed studies of the cases are in preparation. We can conclude by saying that some analysis in physics that neglect Model Theory perspectives are at best approximate.

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REFERENCES

[1] C. Isham, J. Butterfield, *Int. J. Theor. Phys.* **38**, 827 (1999).

[2] C. Isham, *Found. Phys.* **30**, 1707 (2000).

[3] S. Kochen, E.P. Specker, *J. Math. Mechanics* **17**, (1) (1967); *Symp. on the Theory of Models*, North Holland Publ. Co., Amsterdam 1965.

[4] M. Makkai, G.E. Reyes, *First Order Categorical Logic. Model Theoretic Methods in the Theory of topoi and Related Categories., Lect. Notes in Math.*, eds. A. Dold, B. Eckermann, Springer-Verlag, Berlin, Heildelberg 1977.

[5] A. Kock, *Synthethic Differential Geometry., London Math. Sci. Lecture. Notes Series* **51**, Cambridge U. Press, 1981.

[6] I. Moerdijk, G.E. Reyes, *Models for Smooth Infinitesimal Analysis*, Springer Verlag, NY 1991.

[7] S. Mac Lane, *Categories for the Working Mathematician*, Springer Verlag, NY, Heildelberg, Berlin 1971.

[8] A. Tarski, A. Mostowski, R.M. Robinson, *Undecidable Theories*, North Holland Pub. Co., Amsterdam 1953.

[9] K.J. Keisler, C.C. Chang, *Model Theory*, North Holland Publ. Co., Amsterdam 1973.

[10] T. Jech, *Set Theory*, Academic Press, 1978.

[11] R. Murawski, *Funkcje Rekurencyjne i Elementy Metamatematyki*, Uniwersytet im. A. Mickiewicza w Poznaniu, Poznań 1991, in Polish.

[12] Y.B. Grinkevich, `gr-qc/9608013`.

[13] Y.B. Grinkevich, A. Guts, `gr-qc/9610073`.

[14] A. Guts, A. Zvyagitser, `physics/9909016`, `gr-qc/0001076`.

[15] T. Kobayashi, *Classical Solutions of Schroedinger Equations and Non-Standard Analysis* in *Symmetries in Science VII*, eds B. Gruber, T. Otsuka, Plenum Press, NY 1994, pp. 287–294.

[16] T. Kobayashi, *Decoherence Mechanism in QM., Proc of the Symp. of Quantum Informations and Open Systems*, ed. M. Okya, Tokyo Rika University Press, 1 (1995).

[17] A. Robinson, *Collected Papers*, (A. Robinson, P.J. Keleman) pp. 270-278

[18] A. Robinson, *Non-Standard Analysis*, Studies in Logic and the Foundations of Mathematics, Amsterdam 1966.

[19] L. Crane, *J. Math. Phys.* **36**, 6180 (1995).

[20] L. Crane, `gr-qc/0004043`.

[21] L. Crane, D. Yetter, *A Categorical Construction of 4d TQFT* in *Quantum Topology*, eds L.H. Kauffman, R.A. Badhio, World Sci. Press, 1993.

[22] L. Crane, D. Yetter, H. Kauffman, *J. Knot Theory and Its Ramification* **6**, 177 (1997).

[23] M. Mackaay, `qa/9903003`.

[24] J. Baez, J. Dolan, *Categorification* in *Higher Category Theory*, eds E. Getzler, M. Kapranov, AMS, Providence, 1 (1998).

[25] V.G. Turaev, O.Y. Viro, *Topology* **31**, 865 (1992).

[26] F.W. Lawvere, *Functorial Semantics of Algebraic Theories., Proc. Nat. Acad. Sci., USA* **50**, 869 (1963).

[27] F.W. Lawvere, *Continuously Variable Sets: Algebraic Geometry = Geometric Logic*, in Logic Coll. **73**, North Holland, Bristol 1975 p. 135.

[28] E. Witten, *Commun. Math. Phys.* **121**, 357 (1989).

[29] R. Solovay, S. Tennebaum, *Ann. Math.* **94**, 201 (1971).

[30] A. Scedrov, *Forcing and Classifying Topoi., Memoirs of the AMS* **48**, 295 (1984).

[31] J. Dionne, Thèse de M. Sc., Univ. de Montréal, July 1973.

[32] T. Skolem, *Logico-Combinatorial Investigations in the Satisfiability or Provability of Math. Properties ...*, 1920 in *From Frege to Gödel*, ed. J. van Heijenoort, Harvard U. Press., Cambridge Mass 1967.

[33] Y. Suzuki, G. Wilmers, *Non Standard Models for Set Theory*, Proc. B. Russell Memorial Logic Conf., Uldum (1971), Leeds 1973, pp. 278–314.

[34] L. Henkin, *J. Sym. Logic* **14**, 159 (1949).

[35] L. Henkin, *J. Sym. Logic* **15**, 81 (1950).

[36] J. Barwise *Handbook of Math. Logic.*, North Holland, 1977.

[37] K. Gödel, *On Formally Undecidable Propositions of PM and Related Systems,* 592 (1931), in *From Frege to Gödel*, J. van Heijenoort (ed.), Harvard U. Press., Cambridge Mass. 1967.

[38] P.J. Cohen, *Proc. Nat. Acad. Sci. USA* **50**, 1143 (1963).

[39] M. Dalla Chiara, R. Giuntini, *Quantum Logics.*, in *Handbook of Philosophical Logic*, ed. D. Gabbay, Kluver Dordrecht 2001.

[40] M. Farrukh, *J. Math. Phys.* **16**, 177 (1975).

[41] B. Coecke, I. Stubbe, *Found. Phys. Lett.* **12**, 29 (1999).

[42] P. Johnston, *Lect. Notes Math.*, **753**; *Appl. of Sheaves*, Springer Verlag, 1979, 466.

[43] P. Benioff, `quant-ph/0103078`.

[44] P. Benioff, `quant-ph/0106153`.

[45] H. Putnam, *J. Sym. Logic* **45**, 464 (1980).

[46] J. von Neuman, *On the Introduction of Transfinite Numbers,* 1923, in *From Frege to Gödeled*, J. van Heijenoort (ed.), Harvard U. Press., Cambridge Mass. 1967.

[47] T. Bartoszynski, H. Judah, *Set Theory. On the Structure of the Real Line*, ed. A.K. Peters, Wellesley Mass. 1995.

[48] L. Smolin, *J. Math. Phys.* **36**, (1985).

[49] J. Lambek, P.J. Scott, *Introduction to Higher Order Categorical Logic*, Cambridge U. Press, 1994.

[50] A. Ashtekar, *Phys. Rev. Lett.* **57**, 2244 (1986).

[51] J. von Neuman, G. Birghoff, *Ann. Math.* **37**, 823 (1936).