NEW 4D RESULTS FROM SUPERSTRING THEORY*

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(Received October 20, 2011)

Recent advances in relating superstring theory with dimension 4 via exotic smooth geometries on Euclidean $\mathbb{R}^4$ are reviewed. The string theory backgrounds and some configurations of Neveu–Schwarz and Dirichlet branes describe exotic open 4-smoothness. This serves as a link to 4D physics.

DOI:10.5506/APhysPolB.42.2343
PACS numbers: 02.40.–k, 04.50.Kd, 04.60.–m

1. Introduction

Superstring theory is probably the most advanced and promising approach toward formulating a theory of quantum gravity (QG). However, this is not any complete quantum field theory (QFT) and its consistency requires 10 space-time dimensions. In order to grasp 4-dimensional (4D) physics from 10D superstrings, many techniques were worked out. These are in particular: compactification, flux stabilization, brane configuration model-buildings, brane worlds, holography or AdS/CFT correspondence. However, 4d results obtained by these methods are highly ambiguous. Recent proposition [1,2,3,4] relies on considering string theory as rich mathematics which deals with 4-dimensional physics via special 4-geometry. This geometry is smooth, exists exclusively in dimension 4, and is significant to physics in 4d. We have in mind exotic smooth structures on the simplest 4d topological manifold, *i.e.* $\mathbb{R}^4$. Among all $\mathbb{R}^n$ only $n = 4$ case allows for different smoothings of Euclidean $\mathbb{R}^n$. Such programme seems to be a formidable task, since exotic smooth $\mathbb{R}^4$'s are out of reach for conventional mathematical tools usually applicable in lower or higher dimensions in differential geometry and topology. However, some results have been derived where superstring theory was formulated on backgrounds containing 4-dimensional

* Presented at the XXXV International Conference of Theoretical Physics "Matter to the Deepest", Ustroń, Poland, September 12–18, 2011.

part which is exotic $\mathbb{R}^4$ rather than standard smooth $\mathbb{R}^4$. Moreover, some exceptional properties of these 4 Riemannian geometries were found indicating strong connections with physical theories. Thus from this point of view exotic smooth $\mathbb{R}^4$'s serve as a new important channel for superstring theory leading to 4-dimensional physics.

In the next section, we present how exotic smooth $\mathbb{R}^4$s are involved in string theory backgrounds and depend on the certain configurations of Neveu–Schwarz (NS) and Dirichlet (D) branes. In Sec. 2.2, we address the issue of quantum D-branes and show that even this level of string formalism refers to exotic 4-smoothness. Subsequently in Sec. 3, we show the result of the calculation of some energy spectra of a test particle on space-time where the smooth structure is exotic and briefly comment on physics behind exotic 4-smoothness.

2. String backgrounds and quantum D-branes from 4d smoothness

In classical gravity theory, *i.e.* in General Relativity (GR), the geometry is one of (pseudo)-Riemannian differentiable manifolds. String theory has GR (10D Einstein equations) as its classical gravitational limit. However, in the quantum gravity (QG) limit the space-time geometry should be drastically modified. In string theory, the concept of space-time as a smooth manifold is not valid any longer in general. Rather, we have string backgrounds which are described by 2-dimensional conformal field theory (2d CFT) and σ models in suitable targets. However, these string backgrounds still have well-defined geometric classical limits which appear to be the triples (M, g, B) where, in addition to the pseudo-Riemannian smooth manifold M and metric g, we have B-field, *i.e.* local 2-form on M. Conversely, every full string background, hence a 2d CFT plus σ-model with the target M, can be derived from some limiting classical geometry (M, g, B) [5]. In type II superstring theories, in addition to the metric $G_{\mu\nu}$ and an antisymmetric H-field (three-form $H_{\mu\nu\rho}$ is the strength of the B-field) we have the dilaton Φ. In heterotic strings, we have additionally the gauge field $F^a_{\mu\nu}$. Some backgrounds are exact which means they survive at any order of α' corrections. The existence of such backgrounds is of special and exceptional character — the arguments based on these are universal and strong.

Abelian gerbes are classified by the integral classes of $H^3(M, \mathbb{Z})$. These are geometric objects representing the third cohomologies similarly as complex line bundles represent the second cohomologies from $H^2(M, \mathbb{Z})$. The presence of B-field such that H is the curvature of a gerbe indicates that the correct, semi-classical, geometry for string theory is the one based on Abelian gerbes as supplementing Riemannian geometry [5, 6, 7].

Moreover, supposing dilaton is constant and $F^a_{\mu\nu}$ vanishes, the β-function enforces the background be non-flat unless $H = dB$ is zero. Given S^3 part of the linear dilaton background we have non-trivial H-field on it and in order to avoid anomalies we restrict to the integral case $H^3(S^3, \mathbb{Z})$. These classes, however, are non-trivially generated by exotic $\mathbb{R}^4_k, k \in \mathbb{Z}$. Namely, the following strict relation was proved in [8]:

Theorem 1 (2009, [8]) *Suppose we have the family of exotic $\mathbb{R}^4$'s in the radial family whose members are embedded in standard $\mathbb{R}^4$. There exists a corresponding family of 3-spheres in the boundaries of the Akbulut corks for these exotic $\mathbb{R}^4$'s. Then each exotic $\mathbb{R}^4$ from the family as above is determined by the codimension-1 foliation of the corresponding 3-sphere, with non-vanishing Godbillon–Vey (GV) class in $H^3(S^3, \mathbb{R})$. The radius in the family, ρ, and value of GV are related by $GV = \rho^2$.*

On the other hand, the classification of D-branes in string backgrounds is governed by K-theory of the background, or in the presence of H-field, by twisted by H K-theory classes. This is briefly summarized in the next subsections.

2.1. NS and D branes in type II

Let us consider following [2], the bosonic $\mathrm{SU}(2)_k$ WZW model. In the semi-classical limit, *i.e.* $k \to \infty$, D-branes in group manifold SU(2) are determined by wrapping the conjugacy classes of SU(2), *i.e.* are described by 2-spheres S^2s and two poles (degenerate branes) each localized at a point. There are $k+1$ D-branes on the level k SU(2) WZW model [9, 10, 11]. To grasp the dynamics of the branes one should deal with the gauge theory on the stack of N D-branes on S^3. Non-commutative gauge theory emerges [12] similar to the flat space case. Let J be the representation of $\mathrm{SU}(2)_k$ *i.e.* $J = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2}$. The non-commutative action for the dynamics of N branes of type J in the string regime (k is finite), is then given by

$$S_{N,J} = S_{\mathrm{YM}} + S_{\mathrm{CS}} = \frac{\pi^2}{k^2(2J+1)N}\left(\frac{1}{4}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{i}{2}\mathrm{tr}(f^{\mu\nu\rho}\mathrm{CS}_{\mu\nu\rho})\right). \quad (1)$$

Here, the curvature form $F_{\mu\nu}(A) = iL_\mu A_\nu - iL_\nu A_\mu + i[A_\mu, A_\nu] + f_{\mu\nu\rho}A^\rho$ and the non-commutative Chern–Simons action reads $\mathrm{CS}_{\mu\nu\rho}(A) = L_\mu A_\nu A_\rho + \frac{1}{3}A_\mu[A_\nu, A_\rho]$. The fields A_μ, $\mu = 1, 2, 3$ are defined on a fuzzy 2-sphere S^2_J and should be considered as $N \times N$ matrix-valued, *i.e.* $A_\mu = \sum_{j,a} \mathrm{a}^\mu_{j,a} Y^j_a$ where Y^j_a are fuzzy spherical harmonics and $\mathrm{a}^\mu_{j,a}$ are Chan–Paton matrix-valued coefficients. L_μ are generators of the rotations on fuzzy 2-spheres and they act only on fuzzy spherical harmonics [11]. Originally, the action (1) was designed to describe Maxwell theory on fuzzy spheres [13]. The equations of motion derived from (1) read

$$L_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0\,. \tag{2}$$

The solutions of (2) describe the dynamics of the branes, *i.e.* the condensation processes on the brane configuration (N, J) which results in another configuration (N', J'). A special class of solutions, in the semi-classical $k \to \infty$ limit, can be obtained as: for $J = 0$ one has N branes of type $J = 0$, *i.e.* N point-like branes in S^3 sitting at the identity of the group. Given another solution corresponding to $J_N = \frac{N-1}{2}$, then this is the condensed state of N point-like branes at the identity of SU(2) [11]

$$(N, J) = (N, 0) \to \left(1, \frac{N-1}{2}\right) = \left(N', J'\right)\,. \tag{3}$$

Taking k finite one refers to the boundary CFT. It follows that there exists a continuous passage between the partition functions governed by the Verlinde fusion rules coefficients $N^{\,l}_{J_N j}$, which is $N\chi_j(q)$, and the sum of characters $\sum_j N^{\,l}_{J_N j}\chi_l(q)$, where $N = 2J_N + 1$. In the case of N point-like branes, one can determine the decay product of these

$$\begin{aligned} Z_{(N,0)}(q) &\;\to\; Z_{(1,J_N)}\,, \\ (N,0) &\;\to\; (1, J_N) \end{aligned} \tag{4}$$

which extends the similar process at the semi-classical $k \to \infty$ limit (3).

Thus there are $k+1$ stable branes wrapping the conjugacy classes numbered by $J = 0, \frac{1}{2}, ..., \frac{k}{2}$. Placing N point-like branes (each charged by the unit 1) at the pole e causes their decay to the spherical brane J_N wrapping the conjugacy class. Taking more point-like branes to the stack at e gives the more distant S^2 branes until reaching the opposite pole $-e$, where we have single point-like brane with the opposite charge -1. Having identified $k+1$ units of the charge with -1 we obtain the group of RR charges as $\mathbb{Z}_{k+2}$. In the case of SU(2), we get (for $K = k + 2$)

$$K^\star_H\left(S^3\right) = \mathbb{Z}_K\,. \tag{5}$$

Now, let us place this $S^3 \simeq \mathrm{SU}(2)$ at the boundary of the Akbulut cork for some exotic smooth $\mathbb{R}^4_k$ from the radial family as in Theorem 1. The S^3 belongs to the family of 3-spheres as appearing in Theorem 1. Thus the result follows:

Certain small exotic $\mathbb{R}^4$'s generate the group of RR charges of D-branes in the curved background of $S^3 \subset \mathbb{R}^4$.

And the important correspondence follows:

Theorem 2 ([2]) *The classification of RR charges of the branes on the background given by the group manifold $SU(2)$ at the level k (hence the dynamics of D-branes in S^3 in stringy regime) is correlated with the exotic smoothness on $\mathbb{R}^4$ containing this $S^3 = SU(2)$ as the part of the boundary of the Akbulut cork.*

Let us consider now the linear dilaton geometry as the near horizon geometry of the stack of N NS5-branes in *s*upersymmetric model, *i.e.* $\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times \mathrm{SU}(2)_k$. Again placing this $S^3 \simeq \mathrm{SU}(2)$ at the boundary of the cork we have:

Theorem 3 ([2]) *In the geometry of the stack of NS5-branes in type II superstring theories, adding or subtracting a NS5-brane is correlated with the change of the smoothness structure on the transversal $\mathbb{R}^4$.*

2.2. Quantum and topological D-branes

In semi-classical regime, the space-time pseudo-Riemannian manifold in superstring theory is replaced by the geometry of the background generated by B-field. In the quantum regime further modification is required, namely the geometry becomes non-commutative geometry generated by *spectral triples*. Similar change is expected concerning the geometry assigned to the world-volumes of D-branes considered as quantum objects. Thus one defines ([14]) a space-time as some *separable* $C^\star$-algebra $\mathcal{A}$ and D-brane in such space-time is also a separable $C^\star$-algebra given by some spectral triple $(\mathcal{H}, \mathcal{A}_M, T)$. $\mathcal{H}$ is the separable Hilbert space and T is a self-adjoint (unbounded) operator acting on the Chan–Paton Hilbert space.

Every small exotic $\mathbb{R}^4$ from the radial family, as in Theorem 1, determines the codimension-1 foliation of S^3 so one assigns a $C^\star$-algebra to this 4-exotic. This algebra is precisely the non-commutative convolution $C^\star$-algebra $C^\star(V, F)$ of the foliation (V, F). The class of generalized stable D-branes in $C^\star$-algebra is then defined [4]. Interesting connections of the formalism with exotic 4-smoothness emerge:

Theorem 4 (2011, [4]) *The class of generalized stable D-branes on the $C^\star$ algebra $C^\star(S^3, F_1)$ (of the codimension-1 foliation of S^3) determines an invariant of exotic smooth $\mathbb{R}^4$,*

Theorem 5 (2011, [4]) *Let e be an exotic $\mathbb{R}^4$ corresponding to the codimension-1 foliation of S^3 which gives rise to the $C^\star$algebra $\mathcal{A}_e$. The exotic smooth $\mathbb{R}^4$ embedded in e determines a generalized quantum D-brane in $\mathcal{A}_e$.*

Interestingly, the interpretation of D-branes as subspaces can be recovered partially in the quantum regime for the special class of the topological quantum D-branes. The embedding becomes now the wild embedding into space-time, known from the topology of *horned Alexander's spheres*.

Theorem 6 (2011, [3]) *Let $\mathbb{R}^4_H$ be some exotic $\mathbb{R}^4$ determined by an element in $H^3(S^3,\mathbb{R})$, i.e. by a codimension-1 foliation of S^3. Each wild embedding $i : K^3 \to S^p$ for $p > 6$ of a 3-dimensional polyhedron determines a class in $H^n(S^n,\mathbb{R})$ which represents a wild embedding $i : K^p \to S^n$ of a p-polyhedron into S^n.*

Topological quantum Dp-branes are these branes which are determined by the wild embeddings $i : K^p \to S^n$ as above and in the classical and flat limit correspond to tame embeddings. In fact, B-field on S^3 can be translated into wild embeddings of higher dimensional objects and generates quantum character of these branes.

3. Where is physics?

Given the above rather formal results we are going to argue that exotic $\mathbb{R}^4$s are also sound from the point of view of physics in 4D. First, the interesting mixture of string theory backgrounds, CFT and differential geometry techniques, led to the determining the energy spectra of a charged particle, when the particle travels through the space-time and the smooth structure of this space-time is the one of exotic $\mathbb{R}^4$. The calculations are performed under presence of the almost constant magnetic field and in the regime of QG backreactions. The result is the following spectrum [15]

$$\Delta E^k_{j,m,\overline{m}} = \frac{1}{k+2}\left[j(j+1)-m^2\right] + \frac{\left(2\sqrt{k+2}eH - \left(\lambda+\frac{1}{\lambda}\right)m - \left(\lambda-\frac{1}{\lambda}\right)\sqrt{(1+2/k)}\overline{m}\right)^2}{4(k+2)\left(1-2H^2\right)}\,. \quad (6)$$

k labels effects from different exotic $\mathbb{R}^4_k$ and is the square of the quantum radii of SU(2), λ is the moduli due to the gravitational backreaction of the magnetic field H, e is the charge of a particle, $j, m, \overline{m}$ are quantum numbers due to the symmetry of the contracted exotic $\mathbb{R}^4_k$.

Second, exotic $\mathbb{R}^4_k$ regions in 4-space-time act as magnetic monopoles, namely [8]: *Some small, exotic smooth structures on $\mathbb{R}^4$ can act as sources of magnetic field, i.e. monopoles, in space-time. Electric charge in space-time has to be quantized, provided some region has this small exotic smoothness.*

Third, quantum effective matter as in the Kondo effect deals with exotic smooth 4-geometry which when in high temperatures naturally allows for QG description via superstring theory [15].

Next, in Riemannian geometry and GR we have the result: *some large exotic smooth R^4s can act as the external sources of gravitational field in space-time.* This Brans conjecture was proved by Asselmeyer [16] in the compact case, and by Sładkowski [17] in the non-compact case.

Other applications deal with neutrino oscillations and transport in dense neutron stars where exotic 4-geometry is generated naturally, or, fractional quantum Hall effect which refers to BCFT. The work is in progress.

All the results presented here were established in the cooperation with Torsten Asselmeyer-Maluga.

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