SPACETIME STRUCTURE AND UNIFICATION OF FUNDAMENTAL INTERACTIONS*

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I discuss the problem to what extent fundamental interactions determine the structure of spacetime. I show that when we are using only topological methods the spacetime should be modelled on an $\boldsymbol{R}$-compact space. Demanding the existence of a differential structure substantially narrows the choice of possible models but the differential structure may not be unique. I also show by using the noncommutative geometry construction of the standard model that fundamental interactions determine the spacetime in the class of $\boldsymbol{R}$-compact spaces. Fermions are essential for the process of determining the spacetime structure.

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1. Introduction

The outcomes of physical measurements are expressed in rational numbers. Nevertheless we believe that all possible values of physical variables constitute the set of real numbers $\boldsymbol{R}$. It is an idealized view since all measurements are performed with certain accuracy and it is even hard to imagine how can they give irrational numbers. Most of physical theories, including quantum gravity, make use of the notion of spacetime, at least approximately. Therefore physicists spend a lot of time on revealing the origin and the structure of spacetime. The algebra of real continuous functions $C(M)$ on the spacetime manifold M seems to be the key to the whole affair of determining M. This algebra play central rôle in classical and quantum physics, although this fact is not always perceived. Here I would like to analyse how faithful our theoretical models of the spacetime can be. I will

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try to be model independent and avoid unnecessary assumptions. Nevertheless, I will suppose that it is possible to determine the algebra $C(M)$ on the spacetime (assumed to be a topological space) with sufficient for our aim accuracy. This does not mean that we have to be able to find each element of $C(M)$ by direct measurement: some, say, inductive construction should be sufficient. I will call elements of $C(M)$ observables. I will also make use of the algebra of continuous K-valued functions $C(M,K)$, K being a topological ring. Finally, I will show how $C(M,K)$ can be used to construct a field theory of fundamental interactions in the A. Connes' noncommutative geometry formalism and to what extent the spacetime manifold is determined by electroweak interactions.

2. The topology of spacetime

A lot of properties of a topological space M is encoded in the associated algebras $C(M,K)$ of continuous K-valued functions, K being a topological ring, field, algebra etc. Even differential structures on a manifold M can be equivalently defined by appropriate subalgebras $C^k(M,K)$ of real differentiable functions on M. Suppose that our experimental technique is powerful enough to reconstruct $C(M,\boldsymbol{R}) \equiv C(M)$ on our model of the spacetime M. What sort of information concerning M can be extracted from these data? If M is a set and $\mathcal{C}$ a family of real functions $M \to \boldsymbol{R}$ then $\mathcal{C}$ determines a (minimal) topology $\tau_{\mathcal{C}}$ on M such that all function in $\mathcal{C}$ are continuous [1,2]. In general, there will be real continuous functions on M that do not belong to $\mathcal{C}$ and more families of real functions on M would define the same topology on M. So, without loss of generality, we can always suppose that M is a topological space. To be able to distinguish x from y in our model of spacetime we have to find such an observable $f \in C(M)$ that for $x, y \in M \quad f(x) \neq f(y)$. Therefore it seems reasonable to assume that

$$f(x) = f(y) \quad \forall f \in C(M) \quad \Rightarrow \quad x = y\,. \tag{1}$$

From the mathematical point of view, we have to identify all points that are not distinguished by $C(M)$, that is to demand (1). It is then easy to show that such spaces are Hausdorff spaces. This means that we can look for the topological representation of the spacetime in the class of Hausdorff spaces. To proceed let me define [2,4]:

Definition 1. Let E be a topological space. A topological Hausdorff space X is called E-compact (E-regular) if it is homeomorphic to a closed (arbitrary) subspace of some Tychonoff power of E, E^Y.

The following facts justify our assumption (1). For a topological space X, not necessarily a Hausdorff one, we can construct an E-regular space

$\tau_E X$ and its E-compact extension $\upsilon_E X$ so that we have [3,4]

$$C(X,E) \cong C(\tau_E X, E) \cong C(\upsilon_E X, E) \cong C(\upsilon_E \tau_E X, E) , \tag{2}$$

where $\cong$ denotes isomorphism. The spaces $\tau_E X$ and $\upsilon_E \tau_E X$ have the nice property (1). Now, it is obvious that, in general, our theoretical model of the spacetime may not be unique. This important result also says that we can always model our spacetime as a subset of some Tychonoff power of $\boldsymbol{R}$ provided $C(M)$ is known! But it also says that we can model it on a subset of a Tychonoff power of a different topological space *e.g.* the rational numbers $\boldsymbol{Q}$ (*cf.* the discussion at the beginning). So its our choice! The topological number fields $\boldsymbol{R}$ and $\boldsymbol{Q}$ have the additional nice property of determining uniquely (up to a homeomorphism) $\boldsymbol{R}$- and $\boldsymbol{Q}$-compact sets provided the appropriate algebras of continuous functions are known:

$$C(X,E) \cong C(Y,E) \iff X \text{ is homeomorphic to } Y,\ E = \boldsymbol{R} \text{ or } \boldsymbol{Q}\,. \tag{3}$$

Other topological rings can also have this property. But this does not mean that the spacetime modelled on $C(M,E)$ is homeomorphic to the one modelled on $C(M,E')$. Hewitt have shown that $\boldsymbol{R}$-compact spaces are determined up to a homeomorphism by $C(X,E)$, where $E = \boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$ (the topological fields of complex numbers and quaternions, respectively) [5]. This means that if we are interested in modelling spacetime on an $\boldsymbol{R}$-compact space then we can use $C(M,\boldsymbol{R})$, $C(M,\boldsymbol{C})$ or $C(M,\boldsymbol{H})$ to determine it. Such conclusion is false for rational numbers.

Another problem we have to face is to decide if we are dealing with the algebra $C(X,E)$ or only with the algebra of all continuous bounded E-valued functions on X, $C^*(X,E)$ [2,4] if this concept make sense. For a compact space X we have $C(X,E) = C^*(X,E)$, but in general, they are distinct. Spaces on which all continuous real functions are bounded are called pseudocompact. An $\boldsymbol{R}$-compact pseudocompact space is compact. We might get hints that some observables may in fact be unbounded but we are unlikely to be able to "measure infinities". An unbounded observable is necessary to show that the spacetime is a noncompact topological space. If we suppose that we can only recover $C^*(M,\boldsymbol{R}) \equiv C^*(M)$, then we can as well suppose that M is compact (for an $\boldsymbol{R}$-compact M). In general, there will be more spaces with $C^*(M)$ as the algebra of real bounded continuous functions on them (they may not be compact or even $\boldsymbol{R}$-compact). Compactness (or paracompactness) of the space is a welcome property. For example pseudodifferential operators have discrete spectrum on compact spaces. Physicists often compactify configuration spaces by adding extra points or imposing appropriate boundary conditions. Demanding that all physical fields vanish at infinity is usually equivalent to the one point compactification of the spacetime and requiring that all fields vanish at the added "infinity point". In general, a topological space X has more then one compactification. In some sense the one point compactification is minimal and the Stone-Ĉech compactification is maximal [2]. We will probably have to make nontopological assumptions to choose one among the possible compactifications although they can be distinguished by regular subrings of $C(M)$ if they contain constant functions [3, 4].

It may be too optimistic to assume that we are able to determine $C(M, \boldsymbol{R})$ with the required precision. Suppose that our experimental technique allows only for sort of *yes* or *no* answer to questions concerning spacetime structure [6]. In this case we have to consider determination of a topological space X by the ring $C(X, D)$ of continuous functions into $D = \{0, 1\}$ with various topological and/or algebraic structures. In general, $C(X, D)$ does not determine the space X although $C(X, \boldsymbol{Z}_2)$ fulfils (3) with $E = \boldsymbol{Z}_2$. One can also consider other discrete fields *e.g.* $\boldsymbol{Z}_3$ [3, 4]. In such case we can only try to determine the space in the class of E-compact spaces for some discrete E. Topological subfields of $\boldsymbol{R}$ can also be used for that purpose because they fulfil (3) [2–4].

One may also wonder if the knowledge of some symmetries might be of any help. In general, a topological space X is not determined by its symmetries (homeomorphisms $X \to X$) [7, 8] but sometimes can provide us with useful information, *e.g.* if we know that some group G acts transitively on X then the cardinality of X is not greater than the cardinality of G [9]. For example, if we are pretty sure that the Lorentz group acts transitively on the spacetime we have got an upper bound on the cardinality of the spacetime.

Of course, spacetime "points" may have structure that is beyond our experimental scope. This corresponds to determining only some subalgebra of $C(M)$. We have to find a phenomenon that is indescribable in terms of $C(M)$ to reject the assumptions of $\boldsymbol{R}$-compactness of the spacetime.

We do not know if the physical world can be described by using only topological methods. The most spectacular example is the existence of the Whitehead spaces. These are three-dimensional topological manifolds that are not homeomorphic to $\boldsymbol{R}^3$ but their products with $\boldsymbol{R}$ are homeomorphic to $\boldsymbol{R}^4$. In other words when an $\boldsymbol{R}^1$ is factored out in $\boldsymbol{R}^4$ the result will not necessary be $\boldsymbol{R}^3$. One have to demand differentiability for this to be the case. More sophisticated formalism would involve further assumptions about the spacetime structure but it may not be easy to find out if these assumptions are necessary or just convenient tools. I will discuss some aspects of this issue in the following sections.

3. Differential structure?

Differential calculus have proven to be a powerful tool in the hands of physicists. But is it indispensable? Not every topological space or even topological manifold can support differential structures and demanding the existence of a differential structure on the spacetime can severely restrict our choice of spaces for modelling the spacetime. A differential structure on a topological manifold M, if it exists, can be defined by specifying a subalgebra of k-times differentiable functions $C^k(M, \boldsymbol{R})$ of the algebra $C(M)$. The algebra $C^\infty(M)$ of smooth real functions on M determines M up to a diffeomophism [10] (the points of M are in one-to-one correspondence with maximal ideals in $C^\infty(M)$). The algebra of continuous functions on M is larger than $C^k(M, \boldsymbol{R})$ and may correspond to more topological spaces than M but if two manifolds have at some points p and q isomorphic rings of germs of continuous functions then the points p and q have homeomorphic neighbourhoods (local dimensions are the same) [11]. If the laws of physics are "smooth" then the spacetime should be modelled on a smooth manifold. If this is the case then $C^\infty(M, \boldsymbol{R})$ is sufficient to determine M and describe all physical phenomena. Geometrical quantization is one of the most popular efforts in this direction. But in the smooth case we face a new nonuniqueness problem because some manifolds can support many nonequivalent differential structures [12, 19]. Such "additional" differential structures are usually referred to as *fake* or *exotic* ones. They are specially abundant in the four-dimensional case (it is sufficient to remove one point from a given manifold to get a manifold with exotic structures [16]). More astonishing is the fact that the topologically trivial fourdimensional Euclidean space $\boldsymbol{R}^4$ can be given uncountably many exotic structures (in fact a two-parameter family of them) [16]. We have to interpret these mathematical results in physical language [17, 19]. This is not an easy task. Although one can put forward many arguments that exotic smoothness might have physical sense [18], the lack of any tractable (pseudo-) Riemannian structure hinders physical predictions. Nevertheless some problems can be discussed.

4. Noncommutative differential geometry and physical models

As I have noted in the previous section, differential geometry can be formulated in terms of the commutative algebra of real smooth functions on the manifold in question. Connes managed to generalize this result for much larger class of algebras, not necessarily commutative [20, 21]. One should not be surprised that his noncommutative geometry have found profound physical applications. The basic ingredients are a C^*-algebra $\mathcal{A}$ represented in some Hilbert space H and a distinguished operator $\mathcal{D}$ ("Dirac operator") acting in H. The differential da of an $a \in \mathcal{A}$ is defined by $[\mathcal{D}, a]$ and the integral is replaced by the Diximier trace, Tr_ω, with an appropriate inverse n-th power of $|\mathcal{D}|$ instead of the volume element $d^n x$. The Diximier trace of an operator O is roughly speaking the logarithmic divergence of the ordinary trace:

$$\mathrm{Tr}_\omega\, O = \lim_{n\to\infty} \frac{\lambda_1 + \ldots + \lambda_n}{\log\ n}\,, \tag{4}$$

where λ_i is the i-th proper value of O. See [20, 24, 25] for details. One can generalize the notions of covariant derivative (∇), connection (A) and curvature (F) forms so that "standard" properties are conserved:

$$\nabla = d + A\,, \qquad F = \nabla^2 = dA + A^2\,, \tag{5}$$

where $A \in \Omega^1_{\mathcal{D}}$ is the algebra of one forms defined with respect to d [20,21]. Fiber bundles became projective modules on $\mathcal{A}$ in this language. The n-dimensional Yang–Mills fermionic action is given by the formula

$$\mathcal{L}\,(A, \psi, \mathcal{D}) = \mathrm{Tr}_\omega \left(F^2 \mid \mathcal{D} \mid^{-n}\right) + \langle \psi \mid \mathcal{D} + A \mid \psi\rangle\,, \tag{6}$$

where $\langle | \rangle$ denotes the scalar product in the Hilbert space. For $\mathcal{A} = C^\infty(M)$ and $\mathcal{D}$ being the classical Dirac operator we recover the ordinary Riemannian geometry of the spin manifold M. Physicists have learned from the noncommutative geometry that one can describe fundamental interactions by specifying the Hilbert space of fermionic states and a representation of an C^* algebra in this Hilbert space. If one takes

$$\mathcal{A} = C^\infty(M, \boldsymbol{C}) \oplus C^\infty(M, \boldsymbol{H}) \oplus M_{3\times 3}(C^\infty(M, \boldsymbol{C}))\,, \tag{7}$$

the known fermionic states to span the Hilbert space and the generalized Dirac operator including the Kobayashi–Maskawa mass matrix as $\mathcal{D}$ one gets the standard model Lagrangian [20, 23] (I have neglected some important technical details that are not necessary for the present discussion). The structure of the "spacetime algebra" (7) and the analysis given in the previous sections allow us to conclude that the spacetime structure is uniquely determined in the class of $\boldsymbol{R}$-compact spaces by fundamental interactions of fermions (gravitation is hidden in the metric tensor that "enters" the Dirac operator). The knowledge of $C^\infty(M)$ is sufficient for the construction of the manifold M but the Higgs mechanism to be at work requires that M is multiplied by some discrete space [20, 24, 25]. All this means that we may not know the structure of the spacetime with satisfactory precision but nevertheless fundamental interactions determine it in a quite unique way: there is only one spacetime in the class of $\boldsymbol{R}$-compact spaces. It should be noted here that if others rings would appear in (7) then this conclusion may not be true (for example, grand unified models can be less determinative than the "low energy approximation" [23]). Of course, it is still possible that the C^* algebra $\mathcal{A}$ that describes correctly fundamental interactions do not correspond to any topological space. This would mean that spacetime can only approximately be described as a topological space, say, defined by some subalgebra of $\mathcal{A}$ or that fundamental interactions does not determine it uniquely. It should be stressed here that matter fields (fermions) and their interactions are essential in the process determining the spacetime structure (the Dirac operator and the Hilbert space in question). The pure gauge sector is insufficient because two E-compact spaces X and Y are homeomorphic if and only if the categories of all modules over $C(X, E)$ and $C(Y, E)$ are equivalent. The noncommutative geometry formalism even suggest that fermions and their interactions "define" the spacetime via the Dirac operator at least on the theoretical level.

5. Conclusions

I have analysed the problem of determining the spacetime structure. We should be able to determine the spacetime in the class of $\boldsymbol{R}$-compact spaces at least in the abstract sense. We have to find a phenomenon that cannot be described in terms of the algebra $C(M)$ to reject the assumption of $\boldsymbol{R}$-compactness. If we are using only topological methods we will not be able to construct the topological model M of the spacetime uniquely. An unbounded observable is necessary to prove noncompactness of spacetime. In the general case, we will be able to construct only the Stone-Ĉech compactification of the space in question. The existence of a differential structure on M allows for the identification of M with the set of maximal ideals of $C^\infty(M)$, although we anticipate that the determination of the differential structure may be problematic. Connes' construction of the standard model Lagrangian imply that fundamental interactions of matter fields determine the model of spacetime in the class $\boldsymbol{R}$-compact space in a unique way. More general models of fundamental interactions, for example GUTs, are lacking in such a determinative power. Matter fields are essential for defining and determining the spacetime properties. If we are not able to determine $C(M, \boldsymbol{R})$ or $C(M, \boldsymbol{Q})$ then our knowledge of the spacetime structure is substantially limited. If this is the case we have a bigger class of spaces "at our disposal" and we have more freedom in making assumptions about the topology of the spacetime.

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