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RESEARCH ARTICLE

On the center of distances

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Abstract We introduce the notion of a center of distances of a metric space and use it in a generalization of the theorem by John von Neumann on permutations of two sequences with the same set of cluster points in a compact metric space. This notion is also used to study sets of subsums of some sequences of positive reals, as well for some impossibility proofs. We compute the center of distances of the Cantorval, which is the set of subsums of the sequence $\frac{3}{4}, \frac{1}{2}, \frac{3}{16}, \frac{1}{8}, \dots, \frac{3}{4^n}, \frac{2}{4^n}, \dots$, and for other related subsets of the reals.

Keywords Cantorval · Center of distances · von Neumann's theorem · Set of subsums · Digital representation

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1 Introduction

The center of distances seems to be an elementary and natural notion which, as far as we know, has not been studied in the literature. It is an intuitive and natural concept which allows us to prove a generalization of von Neumann's theorem on permutations of two sequences with the same set of cluster points in a compact metric space, see Theorem 2.1. We have realized that the computation of centers of distances—even for well-known metric spaces—is not an easy task because it requires skillful use of fractions. We have only found a few algorithms which enable us to compute centers of distances, see Proposition 3.2 and Lemma 5.1.

We present the use of this notion for impossibility proofs, i.e., to show that a given set cannot be the set of subsums, for example see Corollary 5.5. We refer the readers to the paper [14] by Nitecki, as it provides a good introduction to facts about the set of subsums of a given sequence. It is also worth to look into the papers [1–3,8] as well as others cited therein.

In several papers, the set of all subsums of the sequence $\frac{3}{4}, \frac{1}{2}, \frac{3}{16}, \frac{1}{8}, \ldots, \frac{3}{4^n}, \frac{2}{4^n}, \ldots$, i.e., the set $\mathbb{X}$ consisting of all sums

$$\sum_{n \in A} \frac{2}{4^n} + \sum_{n \in B} \frac{3}{4^n},$$

where A and B are arbitrary subsets of positive natural numbers, is considered. Guthrie and Nymann, see [5] and cf. [15] and [14, p. 865], have shown that $\left[\frac{3}{4}, 1\right] \subset \mathbb{X}$. But, as it can be seen in Corollary 4.2, we get that $\left[\frac{2}{3}, 1\right] \subset \mathbb{X}$. For these reasons, we have an impression that the arithmetical properties of $\mathbb{X}$ are not known well and described in the literature. Results concerning some properties of $\mathbb{X}$ are discussed in Propositions 4.1, 4.3 and 4.4; Corollary 4.5; Theorems 5.2, 5.3, 5.4, 6.1 and 6.2; and they are also presented in Figs. 1, 2 and 3.

2 A generalization of von Neumann's theorem

Given a metric space X with the distance d. Suppose that sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ in X have the same set of cluster points C. For them, von Neumann [13] proved that there exists a permutation $\pi : \omega \to \omega$ such that $\lim_{n\to+\infty} d(x_n, y_{\pi(n)}) = 0$. Proofs of the above statement can be found in [6,18]. However, we would like to present a slight generalization of this result. To prove it we use the so-called "back-and-forth" method, which was developed in [7, pp. 35–36] and is still used successfully by many mathematicians, for example cf. [4,16] or [17], etc. It is also worth mentioning the modern development of classical works of Fraïssé by Kubiś [11].

Consider the set

$$S(X) = \{\alpha : \forall_{x\in X}\, \exists_{y\in X}\, d(x, y) = \alpha\},$$

which will be called the *center of distances* of X.

Theorem 2.1 *Suppose that sequences $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$ in X have the same set of cluster points $C \subseteq X$, where (X, d) is a compact metric space. If $\alpha \in S(C)$, then there exists a permutation $\pi : \omega \to \omega$ such that* $\lim_{n\to+\infty} d(a_n, b_{\pi(n)}) = \alpha$.

Proof Given $\alpha \in S(C)$, we shall renumber $(b_n)_{n\in\omega}$ by establishing a permutation $\pi : \omega \to \omega$ such that

$$\lim_{n\to+\infty} d(a_n, b_{\pi(n)}) = \alpha.$$

Put $\pi(0) = 0$ and assume that values $\pi(0), \pi(1), \dots, \pi(m-1)$ and inverse values $\pi^{-1}(0), \pi^{-1}(1), \dots, \pi^{-1}(m-1)$ are already defined. We proceed step by step as follows.

If $\pi(m)$ is not defined, then take points $x_m, y_m \in C$ such that $d(a_m, x_m) = d(a_m, C)$ and $d(x_m, y_m) = \alpha$. Choose $b_{\pi(m)}$ to be the first element of $(b_n)_{n\in\omega}$ not already used such that $d(y_m, b_{\pi(m)}) < \frac{1}{m}$.

If $\pi^{-1}(m)$ is not defined, then take points $p_m, q_m \in C$ such that $d(b_m, q_m) = d(b_m, C)$ and $d(p_m, q_m) = \alpha$. Choose $a_{\pi^{-1}(m)}$ to be the first element of $(a_n)_{n\in\omega}$ not already used such that $d(p_m, a_{\pi^{-1}(m)}) < \frac{1}{m}$.

The set $C \subseteq X$, as a closed subset of a compact metric space, is compact. Hence the required points x_m, y_m, p_m and q_m always exist and also

$$\lim_{n\to+\infty} d(a_n, C) = 0 = \lim_{n\to+\infty} d(b_n, C).$$

It follows that $\alpha = d(x_m, y_m) = d(p_m, q_m) = \lim_{n\to+\infty} d(a_n, b_{\pi(n)})$. □

Let us note that von Neumann's theorem mentioned above is applicable for some other problems, for example cf. [9] or [10], etc. As we have seen, the notion of a center of distances appears in a natural way in the context of metric spaces. Though the computation of centers of distances is not an easy task, it can be done for important examples giving further information about these objects.

3 On the center of distances and the set of subsums

Given a metric space X, observe that $0 \in S(X)$ and also, if $X \subseteq [0, +\infty)$ and $0 \in X$, then $S(X) \subseteq X$.

If $(a_n)_{n\in\omega}$ is a sequence of reals, then the set

$$X = \left\{ \sum_{n\in A} a_n : A \subseteq \omega \right\}$$

is called the *set of subsums* of $(a_n)_{n\in\omega}$. In this case, we have $d(x, y) = |x - y|$. If X is a subset of the reals, then any maximal interval (α, β) disjoint from X is called an X-*gap*. Additionally, when X is a closed set, then any maximal interval $[\alpha, \beta]$ included in X is called an X-*interval*.

Proposition 3.1 *If X is the set of subsums of a sequence $(a_n)_{n\in\omega}$, then $a_n \in S(X)$, for all $n \in \omega$.*

Proof Suppose $x = \sum_{n\in A} a_n \in X$. If $n \in A$, then $x - a_n \in X$ and $d(x, x - a_n) = a_n$. When $n \notin A$, then $x + a_n \in X$ and $d(x, x + a_n) = a_n$. □

In some cases, the center of distances of the set of subsums of a given sequence can be determined. For example, the unit interval is the set of subsums of the sequence $\left(\frac{1}{2^n}\right)_{n>0}$. So, the center of distances of the subsums of $\left(\frac{1}{2^n}\right)_{n>0}$ is equal to $\left[0, \frac{1}{2}\right]$.

Proposition 3.2 *Assume that $(\lambda\cdot[0, b)) \cap X = \lambda\cdot X$, for a number $\lambda > 0$ and a set $X \subseteq [0, b)$. If $x \in [0, b)\setminus X$ and $n \in \omega$, then $\lambda^n x \notin X$.*

Proof Without loss of generality, assume that $X \cap (0, b) \neq \varnothing$. Thus $\lambda \leqslant 1$, since otherwise we would get $b < \lambda^m t \in X$, for some $t \in X$ and $m \in \omega$. Obviously $x = \lambda^0 x \in [0, b)\setminus X$. Assume that $\lambda^n x \notin X$, so we get

$$\lambda\cdot[0, b) \ni \lambda^{n+1}x = \lambda\cdot\lambda^n x \notin \lambda\cdot X = (\lambda\cdot[0, b)) \cap X.$$

Therefore $\lambda^{n+1}x \notin X$, which completes the induction step. □

Using Proposition 3.2 with $\lambda = \frac{1}{q^n}$ and $b = 1$, one can prove the next theorem. In fact, this proposition explains the hidden argument in the next proof.

Theorem 3.3 *If $q > 2$ and $a > 0$, then the center of distances of the set of subsums of a geometric sequence $\left(\frac{a}{q^n}\right)_{n>0}$ consists of exactly zero and the terms of this sequence.*

Proof By Proposition 3.1, we get $\frac{a}{q^n} \in S(X)$ for $n > 0$. Without loss of generality we can assume $a = 1$. The diameter of the set X of subsums of the sequence $\left(\frac{1}{q^n}\right)_{n>0}$ equals $\frac{1}{q-1} = \sum_{i>0} \frac{1}{q^i} = \mathcal{X}(0)$. Putting $\mathcal{X}(n) = \sum_{i>n} \frac{1}{q^i}$, since $\frac{1}{q-1} < 1$, we get

$$\frac{1}{q^n} > \sum_{i>n} \frac{1}{q^i} = \frac{1}{q^n}\cdot\frac{1}{q-1} = \mathcal{X}(n) > \frac{1}{q^{n+1}}.$$

So, $\frac{1}{q} \in X$ witnesses that no $t > \frac{1}{q}$ belongs to $S(X)$. Indeed,

$$\frac{1}{q} - t < 0 \quad \text{and} \quad \frac{1}{q} + t > \frac{2}{q} > \frac{1}{q-1} = \mathcal{X}(0).$$

If $t \in I$, where I is an X-gap, then $t \notin S(X)$. Indeed, then $t \notin X$ and $-t \notin X$, i.e., $0 \in X$ witnesses that $t \notin S(X)$. But the intervals $\left(\mathcal{X}(n), \frac{1}{q^n}\right)$ are X-gaps, hence $\bigcup_{n>0} \left(\mathcal{X}(n), \frac{1}{q^n}\right)$ is disjoint from $S(X)$.

Now, assume that $n > 0$ is fixed. Suppose that $t \in \left(\frac{1}{q^{n+1}}, \mathcal{X}(n)\right]$ and $t + \mathcal{X}(n+1) < \frac{1}{q^n}$. Thus $\mathcal{X}(n+1) \in X$ witnesses that $t \notin S(X)$. Indeed, $\mathcal{X}(n+1) < \frac{1}{q^{n+1}} < t$ implies $\mathcal{X}(n+1) - t < 0$, and $\mathcal{X}(n+1) + \frac{1}{q^{n+1}} = \mathcal{X}(n)$ implies that the X-gap $\left(\mathcal{X}(n), \frac{1}{q^n}\right)$ has to contain $t + \mathcal{X}(n+1)$.

If $t \in \left(\frac{1}{q^{n+1}}, \mathcal{X}(n)\right]$ and $\frac{1}{q^n} \leqslant t + \mathcal{X}(n+1)$, then the interval $\left(\mathcal{X}(n) - t, \frac{1}{q^n} - t\right)$ is included in the interval $[0, \mathcal{X}(n+1)]$. No X-gap of the length

$$\frac{q-2}{q^n(q-1)} = \frac{1}{q^n} - \mathcal{X}(n)$$

is contained in the interval $[0, \mathcal{X}(n+1)]$. Therefore, one can find

$$x \in X \cap \left(\mathcal{X}(n) - t, \frac{1}{q^n} - t\right) \subset [0, \mathcal{X}(n+1)],$$

which witnesses that $t \notin S(X)$. Indeed, we get $x \leqslant \mathcal{X}(n+1) < \frac{1}{q^{n+1}} < t$, hence $x - t < 0$; and we have that $x + t$ belongs to the X-gap $\left(\mathcal{X}(n), \frac{1}{q^n}\right)$.

By Proposition 3.1, we get $\frac{1}{q^n} \in S(X)$ for $n > 0$. □

Note that, when we put $a = 2$ and $q = 3$, Theorem 3.3 applies to the Cantor ternary set. For $a \in \{2, 3\}$ and $q = 4$ this theorem applies to sets $\mathcal{C}_1$ and $\mathcal{C}_2$ which will be defined in Sect. 4.

4 An example of a Cantorval

Following [5, p. 324], consider the set of subsums

$$\mathbb{X} = \left\{\sum_{n>0} \frac{x_n}{4^n} : \forall_n \; x_n \in \{0, 2, 3, 5\}\right\}.$$

Thus, $\mathbb{X} = \mathcal{C}_1 + \mathcal{C}_2$, where $\mathcal{C}_1 = \left\{\sum_{n\in A} \frac{2}{4^n} : 0 \notin A \subseteq \omega\right\}$ and $\mathcal{C}_2 = \left\{\sum_{n\in B} \frac{3}{4^n} : 0 \notin B \subseteq \omega\right\}$. Following [12, p. 330], because of its topological structure, one can call this set a *Cantorval* (or an $\mathcal{M}$-Cantorval).

Before discussing the affine properties of the Cantorval $\mathbb{X}$ we shall introduce the following useful notions. Every point x in $\mathbb{X}$ is determined by a sequence $(x_n)_{n>0}$, where $x = \sum_{n>0} \frac{x_n}{4^n}$. The value x_n is called the n-th *digit* of x and the sequence $(x_n)_{n>0}$ is called a *digital representation* of the point $x \in \mathbb{X}$. Keeping in mind the formula for the sum of an infinite geometric series, we denote the tails of series as follows:

$$\mathcal{C}_1(n) = \sum_{k>n} \frac{2}{4^k} = \frac{2}{3} \cdot \frac{1}{4^n} \quad \text{and} \quad \mathcal{C}_1(0) = \frac{2}{3};$$

$$\mathcal{C}_2(n) = \sum_{k>n} \frac{3}{4^k} = \frac{1}{4^n} \quad \text{and} \quad \mathcal{C}_2(0) = 1;$$

$$\mathcal{X}(n) = \sum_{k>n} \frac{5}{4^k} = \frac{5}{3} \cdot \frac{1}{4^n} \quad \text{and} \quad \mathcal{X}(0) = \frac{5}{3}.$$

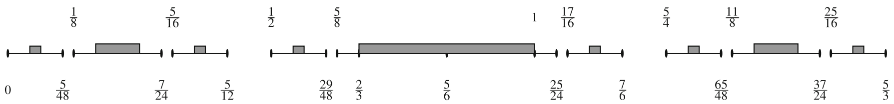

Fig. 1 An approximation of the Cantorval $\mathbb{X} \subset \left[0, \frac{5}{3}\right]$

Since $\mathcal{X}(0) = \frac{5}{3}$ we get $\mathbb{X} \subset \left[0, \frac{5}{3}\right]$. The involution $h \colon \mathbb{X} \to \mathbb{X}$ defined by the formula

$$x \mapsto h(x) = \frac{5}{3} - x$$

is the symmetry of $\mathbb{X}$ with respect to the point $\frac{5}{6}$. In order to check this, it suffices to note that

$$\sum_{n \in A} \frac{2}{4^n} + \sum_{n \in B} \frac{3}{4^n} = x \in \mathbb{X} \quad \Longrightarrow \quad \frac{5}{3} - x = \sum_{\substack{n \notin A \\ n>0}} \frac{2}{4^n} + \sum_{\substack{n \notin B \\ n>0}} \frac{3}{4^n} \in \mathbb{X};$$

and also that

$$\frac{1}{2} + \sum_{n>0} \frac{5}{4^{2n}} = \frac{3}{4} + \sum_{n>0} \frac{5}{4^{2n+1}} = \frac{5}{6} \in \mathbb{X}.$$

So, we get $\mathbb{X} = \frac{5}{3} - \mathbb{X}$ and $\mathbb{X} = h[\mathbb{X}]$.

In Fig. 1, there are marked gaps $\left(\frac{5}{12}, \frac{1}{2}\right)$ and $\left(\frac{7}{6}, \frac{5}{4}\right)$, both of the length $\frac{1}{12}$. Six gaps $\left(\frac{5}{48}, \frac{1}{8}\right)$, $\left(\frac{7}{24}, \frac{5}{16}\right)$, $\left(\frac{29}{48}, \frac{5}{8}\right)$, $\left(\frac{25}{24}, \frac{17}{16}\right)$, $\left(\frac{65}{48}, \frac{11}{8}\right)$ and $\left(\frac{37}{24}, \frac{25}{16}\right)$ have the length $\frac{1}{48}$. The rest of gaps are shorter and have lengths not greater than $\frac{1}{192}$. To describe intervals which lie in $\mathbb{X}$, we need the following. Let

$$K_n = \left[\frac{2}{3}, 1\right] \cap \left\{ \sum_{i=1}^{n} \frac{x_i}{4^i} : \forall_i \; x_i \in \{0, 2, 3, 5\} \right\}.$$

We get $K_1 = \left\{\frac{3}{4}\right\}$ and $K_2 = \left\{\frac{11}{16}, \frac{3}{4}, \frac{13}{16}, \frac{7}{8}, \frac{15}{16}\right\}$. Keeping in mind $\mathcal{C}_1(0) = \frac{2}{3}$ and $\mathcal{C}_1(n) < \frac{1}{4^n}$, we check that $f_1^n = \frac{3}{4^n} + \sum_{i=1}^{n-1} \frac{2}{4^i} > \frac{2}{3}$ is the smallest real number in K_n. Similarly, using $\mathcal{X}(n) = \frac{5}{3} \cdot \frac{1}{4^n} < \frac{2}{4^n}$ and $\mathcal{C}_2(0) = 1$, we check that $f_{|K_n|}^n = \sum_{i=1}^{n} \frac{3}{4^i} < 1$ is the greatest real number in K_n. In fact, we have the following.

Proposition 4.1 *Reals from K_n are distributed consecutively at the distance $\frac{1}{4^n}$, from $\frac{3}{4^n} + \sum_{i=1}^{n-1} \frac{2}{4^i}$ up to $\sum_{i=1}^{n} \frac{3}{4^i}$, in the interval $\left[\frac{2}{3}, 1\right]$. Therefore $|K_n| = \frac{1}{3}(4^n - 1)$ and $|K_{n+1}| = 4|K_n| + 1$.*

Proof Since $|K_1| = 1$ and $|K_2| = 5$, the assertions are correct in these cases. Suppose that $K_{n-1} = \{f_1^{n-1}, f_2^{n-1}, \dots, f_{|K_{n-1}|}^{n-1}\}$, where

$$f_1^{n-1} = \frac{3}{4^{n-1}} + \sum_{i=1}^{n-2} \frac{2}{4^i} \quad \text{and} \quad f_{j+1}^{n-1} - f_j^{n-1} = \frac{1}{4^{n-1}}$$

for $0 < j < |K_{n-1}| - 1$; in consequence $f_{|K_{n-1}|}^{n-1} = \sum_{i=1}^{n-1} \frac{3}{4^i}$. Consider the sum

$$K_{n-1} \cup \left(\frac{2}{4^n} + K_{n-1}\right) \cup \left(\frac{3}{4^n} + K_{n-1}\right) \cup \left(\frac{5}{4^n} + K_{n-1}\right),$$

next remove the point $\frac{5}{4^n} + \sum_{i=1}^{n-1} \frac{3}{4^i} > 1$, and then add points $f_1^n = \frac{3}{4^n} + \sum_{i=1}^{n-1} \frac{2}{4^i}$ and $f_3^n = \frac{5}{4^n} + \sum_{i=1}^{n-1} \frac{2}{4^i}$. We obtain the set

$$K_n = \left\{f_1^n, f_2^n, \dots, f_{\frac{1}{3}(4^n-1)}^n\right\},$$

which is what we need. □

Corollary 4.2 *The interval $\left[\frac{2}{3}, 1\right]$ is included in the Cantorval $\mathbb{X}$.*

Proof The union $\bigcup\{K_n : n > 0\}$ is dense in the interval $\left[\frac{2}{3}, 1\right]$. □

Note that it has been observed that $\left[\frac{3}{4}, 1\right] \subset \mathbb{X}$, see [5] or cf. [14]. Since $\mathbb{X}$ is centrally symmetric with $\frac{5}{6}$ as a point of inversion, this yields another proof of the above corollary. However, our proof seems to be new and it is different from the one included in [5].

Put $C_n = \frac{1}{4^n} \cdot \mathbb{X} = \mathbb{X} \cap \left[0, \frac{5}{3\cdot 4^n}\right]$, for $n \in \omega$. So, each C_n is an affine copy of $\mathbb{X}$.

Proposition 4.3 *The subset $\mathbb{X}\setminus\left[\frac{2}{3}, 1\right] \subset \mathbb{X}$ is the union of pairwise disjoint affine copies of $\mathbb{X}$. In particular, this union includes two isometric copies of $C_n = \frac{1}{4^n} \cdot \mathbb{X}$, for every $n > 0$.*

Proof The desired affine copies of $\mathbb{X}$ are C_1 and $\frac{5}{4} + C_1 = h[C_1]$, $\frac{1}{2} + C_2$ and $h\left[\frac{1}{2} + C_2\right]$, and so on, i.e., $\sum_{i=1}^{n} \frac{2}{4^n} + C_{n+1}$ and $h\left[\sum_{i=1}^{n} \frac{2}{4^n} + C_{n+1}\right]$. □

Proposition 4.4 *The subset $\mathbb{X}\setminus\left(\left(\frac{2}{3}, 1\right) \cup \left(\frac{1}{6}, \frac{1}{4}\right) \cup \left(\frac{17}{12}, \frac{3}{2}\right)\right) \subset \mathbb{X}$ is the union of six pairwise disjoint affine copies of $D = \left[0, \frac{1}{6}\right] \cap \mathbb{X}$.*

Proof The desired affine copies of $D = \frac{1}{4} \cdot \left(\mathbb{X} \cap \left[0, \frac{2}{3}\right]\right)$ lie as shown in Fig. 2. □

Fig. 2 The arrangement of affine copies of D

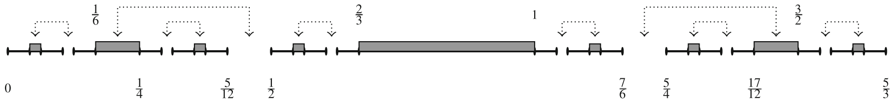

Fig. 3 The correspondence between $\mathbb{X}$-gaps and $\mathbb{X}$-intervals

Corollary 4.5 *The Cantorval* $\mathbb{X} \subset \left[0, \frac{5}{3}\right]$ *has Lebesgue measure* 1.

Proof There exists a one-to-one correspondence between $\mathbb{X}$-gaps and $\mathbb{X}$-intervals as it is shown in Fig. 3.

In view of Propositions 4.3 and 4.4, we calculate the sum of lengths of all gaps which lie in $\left[0, \frac{5}{3}\right] \setminus \mathbb{X}$ as follows:

$$\frac{1}{6} + 6 \cdot \frac{1}{3 \cdot 4^2} + \frac{1}{8} \cdot \frac{3}{4} + \cdots + \frac{1}{8} \cdot \left(\frac{3}{4}\right)^n + \cdots = \frac{1}{6} + \frac{1}{8} \sum_{n \geqslant 0} \left(\frac{3}{4}\right)^n = \frac{2}{3}.$$

Since $\frac{5}{3} - \frac{2}{3} = 1$ we are done. $\square$

If we remove the longest interval from $\frac{1}{4^n} \cdot D$, then we get the union of three copies of D, each congruent to $\frac{1}{4^{n+1}} \cdot D$. This observation—we used it above by default—is sufficient to calculate the sum of lengths of all $\mathbb{X}$-intervals as follows:

$$\frac{1}{3} + \frac{1}{6} + 6 \cdot \frac{1}{3 \cdot 4^2} + \cdots + \frac{1}{8} \cdot \left(\frac{3}{4}\right)^n + \cdots = 1.$$

Therefore the boundary $\mathbb{X} \setminus \operatorname{Int} \mathbb{X}$ is a null set.

5 Computing centers of distances

In case of subsets of the real line we formulate the following lemma.

Lemma 5.1 *Given a set* $C \subseteq [0, +\infty)$ *disjoint from an interval* (α, β)*, assume that* $x \in \left[0, \frac{\alpha}{2}\right] \cap C$*. Then the center of distances* $S(C)$ *is disjoint from the interval* $(\alpha - x, \beta - x)$*, i.e.,* $S(C) \cap ((\alpha, \beta) - x) = \varnothing$*.*

Proof Given $x \in \left[0, \frac{\alpha}{2}\right] \cap C$, consider $t \in (\alpha - x, \beta - x)$. We get

$$x \leqslant \frac{\alpha}{2} \leqslant \alpha - x < t < \beta - x.$$

Since $\alpha < x + t < \beta$, we get $x + t \notin C$, also $x < t$ implies $x - t \notin C$. Therefore $x \in C$ witnesses $t \notin S(C)$. $\square$

We will apply the above lemma by putting suitable C-gaps in the place of the interval (α, β). In order to obtain $t \notin S(C)$, we must find $x < t$ such that $x + t \in (\alpha, \beta)$ and $x \in C$. For example, this is possible when $\frac{\alpha}{2} < t < \alpha$ and the interval $[0, \alpha]$ includes no C-gap of the length greater than or equal to $\beta - \alpha$. But if such a gap exists, then we choose the required x more carefully.

Theorem 5.2 *The center of distances of the Cantorval* $\mathbb{X}$ *is equal to* $\left\{0, \frac{3}{4}, \frac{1}{2}, \ldots, \frac{3}{4^n}, \frac{2}{4^n}, \ldots\right\}$.

Proof The diameter of $\mathbb{X}$ is $\frac{5}{3}$ and $\frac{5}{6} \in \mathbb{X}$, hence no $t > \frac{5}{6}$ belongs to $S(\mathbb{X})$. We use Lemma 5.1 with respect to the gap $(\alpha, \beta) = \left(\frac{7}{6}, \frac{5}{4}\right)$. Keeping in mind the affine description of $\mathbb{X}$, we see that the set $\mathbb{X} \cap \left[0, \frac{7}{12}\right]$ has a gap $\left(\frac{5}{12}, \frac{1}{2}\right)$ of the length $\frac{1}{12}$. For $t \in \left(\frac{7}{12}, \frac{7}{6}\right) \setminus \left\{\frac{3}{4}\right\}$, we choose x in $\mathbb{X}$ such that $x \in \left(\frac{7}{6} - t, \frac{5}{4} - t\right)$. So, if $t \in \left(\frac{7}{12}, \frac{7}{6}\right) \setminus \left\{\frac{3}{4}\right\}$, then $t \notin S(\mathbb{X})$. Similarly using Lemma 5.1 with the gap $(\alpha, \beta) = \left(\frac{29}{48}, \frac{5}{8}\right)$, we check that for $t \in \left(\frac{29}{96}, \frac{7}{12}\right] \setminus \left\{\frac{1}{2}\right\}$ there exists x in $\mathbb{X}$ such that $x \in \left(\frac{29}{48} - t, \frac{5}{8} - t\right)$. Hence, if $t \in \left(\frac{29}{96}, \frac{7}{12}\right] \setminus \left\{\frac{1}{2}\right\}$, then $t \notin S(\mathbb{X})$. Analogously, using Lemma 5.1 with the gap $(\alpha, \beta) = \left(\frac{5}{12}, \frac{1}{2}\right)$, we check that if $\frac{5}{24} < t \leqslant \frac{29}{96} < \frac{5}{12}$, then $t \notin S(\mathbb{X})$.

For the remaining part of the interval $[0, +\infty)$ the proof uses the similarity of $\mathbb{X}$ with $\frac{1}{4^n} \cdot \mathbb{X}$ for $n > 0$. Indeed, we have shown that the $\mathbb{X}$-gaps $\left(\frac{7}{6}, \frac{5}{4}\right)$, $\left(\frac{29}{48}, \frac{5}{8}\right)$ and $\left(\frac{5}{12}, \frac{1}{2}\right)$ witness that $S(\mathbb{X}) \cap \left(\frac{5}{24}, \frac{7}{6}\right) = \left\{\frac{1}{2}, \frac{3}{4}\right\}$. For $n > 0$, by the similarity, the $\mathbb{X}$-gaps $\frac{1}{4^n} \cdot \left(\frac{7}{6}, \frac{5}{4}\right)$, $\frac{1}{4^n} \cdot \left(\frac{29}{48}, \frac{5}{8}\right)$ and $\frac{1}{4^n} \cdot \left(\frac{5}{12}, \frac{1}{2}\right)$ witness that $S(\mathbb{X}) \cap \left(\frac{5}{6 \cdot 4^{n+1}}, \frac{7}{6 \cdot 4^n}\right) = \left\{\frac{2}{4^{n+1}}, \frac{3}{4^{n+1}}\right\}$.

We have $\left\{\frac{2}{4^n} : n > 0\right\} \cup \left\{\frac{3}{4^n} : n > 0\right\} \subseteq S(\mathbb{X})$ by Proposition 3.1. $\square$

Denote $\mathbb{Z} = \left[0, \frac{5}{3}\right] \setminus \operatorname{Int} \mathbb{X}$. Thus the closure of an $\mathbb{X}$-gap is a $\mathbb{Z}$-interval and the interior of an $\mathbb{X}$-interval is a $\mathbb{Z}$-gap.

Theorem 5.3 *The center of distances of the set* $\mathbb{Z}$ *is trivial, i.e.,* $S(\mathbb{Z}) = \{0\}$.

Proof If $\alpha > 1$, then $\{1 + \alpha, 1 - \alpha\} \cap \mathbb{Z} = \varnothing$, hence $1 \in \mathbb{Z}$ implies $\alpha \notin S(\mathbb{Z})$. If $\alpha \in \left\{\frac{1}{4}, 1\right\}$, then $\frac{11}{24} \in \left(\frac{5}{12}, \frac{1}{2}\right) \subset \mathbb{Z}$ implies $\alpha \notin S(\mathbb{Z})$. Indeed, the number $1 + \frac{11}{24}$ belongs to the $\mathbb{Z}$-gap $\left(\frac{17}{12}, \frac{3}{2}\right)$ and the number $\frac{1}{4} + \frac{11}{24}$ belongs to the $\mathbb{Z}$-gap $\left(\frac{2}{3}, 1\right)$ and the number $\frac{11}{24} - \frac{1}{4}$ belongs to the $\mathbb{Z}$-gap $\left(\frac{1}{6}, \frac{1}{4}\right)$. Also $0 \in \mathbb{Z}$ implies $\left(\frac{2}{3}, 1\right) \cap S(\mathbb{Z}) = \varnothing$, since $\left(\frac{2}{3}, 1\right)$ is a $\mathbb{Z}$-gap. For the same reason $\frac{1}{4} \in \mathbb{Z}$ implies $\frac{2}{3} \notin S(\mathbb{Z})$ and $\frac{1}{3} \in \mathbb{Z}$ implies that no $\alpha \in \left(\frac{1}{3}, \frac{2}{3}\right)$ belongs to $S(\mathbb{Z})$. Since $\frac{1}{6} < \frac{17}{32} - \frac{1}{3} < \frac{1}{4}$ and $\frac{2}{3} < \frac{17}{32} + \frac{1}{3} < 1$, then $\frac{17}{32} \in \mathbb{Z}$ implies $\frac{1}{3} \notin S(\mathbb{Z})$. But, if $\alpha \in \left(\frac{1}{4}, \frac{1}{3}\right)$, then $\frac{1}{2} \in \mathbb{Z}$ implies $\alpha \notin S(\mathbb{Z})$. Indeed, $\frac{1}{6} < \frac{1}{2} - \alpha < \frac{1}{4}$ and $\frac{3}{4} < \frac{1}{2} + \alpha < \frac{5}{6}$. So far, we have shown $S(\mathbb{Z}) \cap \left[\frac{1}{4}, +\infty\right) = \varnothing$. In fact, sets $\left[\frac{1}{4^{n+1}}, \frac{1}{4^n}\right]$ and $S(\mathbb{Z})$ are always disjoint, since

$$\frac{1}{4^n} \cdot (\mathbb{Z} \cap [0, 1]) = \left[0, \frac{1}{4^n}\right] \cap \mathbb{Z}.$$

Therefore $\alpha \in \left[\frac{1}{4^{n+1}}, \frac{1}{4^n}\right] \cap S(\mathbb{Z})$ implies $4^n \cdot \alpha \in \left[\frac{1}{4}, 1\right] \cap S(\mathbb{Z})$, a contradiction. Finally, we get $S(\mathbb{Z}) = \{0\}$. $\square$

Now, denote $\mathbb{Y} = \mathbb{Z} \cap \mathbb{X} = \mathbb{X} \setminus \operatorname{Int} \mathbb{X}$. Thus, each $\mathbb{X}$-gap is also a $\mathbb{Y}$-gap, and the interior of an $\mathbb{X}$-interval is a $\mathbb{Y}$-gap.

Theorem 5.4 $S(\mathbb{Y}) = \{0\} \cup \left\{\frac{1}{4^n} : n \in \omega\right\}$.

Proof Since the numbers $0, \frac{1}{4}, \frac{1}{2}, \frac{17}{32}, 1$ and $\frac{1}{3} = \sum_{n>0} \frac{5}{4^{2n}}$ are in $\mathbb{Y}$, we get

$$\bigcup \left\{ \left(\frac{1}{4^{n+1}}, \frac{1}{4^n} \right) : n \in \omega \right\} \cap S(\mathbb{Y}) = \varnothing,$$

as in the proof of Theorem 5.3. We see that $1 \in S(\mathbb{Y})$, because

$$\mathbb{Y} \cap \left[0, \frac{2}{3}\right] + 1 = \mathbb{Y} \cap \left[1, \frac{5}{3}\right].$$

Moreover

$$\left(\mathbb{Y} \cap \left[0, \frac{1}{6}\right] + \frac{1}{4}\right) \cup \left(\mathbb{Y} \cap \left[0, \frac{1}{6}\right] + \frac{1}{2}\right) \subset \mathbb{Y},$$

so $\frac{1}{4} \in S(\mathbb{Y})$. Similarly, we check that $\frac{1}{4^n} \in S(\mathbb{Y})$. □

Corollary 5.5 *Neither $\mathbb{Z}$ nor $\mathbb{Y}$ is the set of subsums of a sequence.*

Proof Since $S(\mathbb{Z}) = \{0\}$, Proposition 3.1 decides the case of $\mathbb{Z}$. Also, this proposition decides the cases of $\mathbb{Y}$, since $\frac{5}{3} \in \mathbb{Y}$ and $\sum_{n \in \omega} \frac{1}{4^n} = \frac{4}{3}$. □

Let us add that the set of subsums of the sequence $\left(\frac{1}{4^n}\right)_{n \in \omega}$ is included in $\mathbb{Y}$. One can check this, observing that each number $\sum_{n \in A} \frac{1}{4^n}$, where the nonempty set $A \subset \omega$ is finite, is the right end of an $\mathbb{X}$-interval.

6 Digital representation of points in the Cantorval $\mathbb{X}$

Assume that $A = (a_n)_{n>0}$ and $B = (b_n)_{n>0}$ are digital representations of a point $x \in \mathbb{X}$, i.e.,

$$\sum_{n>0} \frac{a_n}{4^n} = \sum_{n>0} \frac{b_n}{4^n} = x,$$

where $a_n, b_n \in \{0, 2, 3, 5\}$. We are going to describe dependencies between a_n and b_n. Suppose n_0 is the least index such that $a_{n_0} \neq b_{n_0}$. Without loss of generality, we can assume that $a_{n_0} = 2 < b_{n_0} = 3$, bearing in mind that $\mathcal{X}(n_0) = \frac{5}{3} \cdot \frac{1}{4^{n_0}}$. And then we say that A is *chasing* B (or B is being *caught* by A) in the n_0-step: in other words, $\sum_{i>n_0} \frac{a_i}{4^i} = \sum_{i>n_0} \frac{b_i}{4^i} + \frac{1}{4^{n_0}}$ and $a_k = b_k$ for $k < n_0$. If it is never the case that $a_k = 5$ and $b_k = 0$, then it has to be $b_k + 3 = a_k$ for all $k > n_0$. In such a case, we obtain $\sum_{i>n_0} \frac{a_i}{4^i} = \sum_{i>n_0} \frac{b_i}{4^i} + \frac{1}{4^{n_0}}$, since $\sum_{i>n_0} \frac{3}{4^i} = \frac{1}{4^{n_0}}$.

Suppose n_1 is the least index such that $a_{n_1} = 5$ and $b_{n_1} = 0$, thus B is chasing A in the n_1-step. Proceeding this way, we obtain an increasing (finite or infinite) sequence $n_0 < n_1 < \cdots$ such that $|a_{n_k} - b_{n_k}| = 5$ and $|a_i - b_i| = 3$ for $n_0 < i \notin \{n_1, n_2, \ldots\}$. Moreover, A starts chasing B in the n_k-step for even k's and B starts chasing A in the n_k-step for odd k's, for the rest of steps changes of chasing do not occur.

Theorem 6.1 *Assume that $x \in \mathbb{X}$ has more than one digital representation. There exist a finite or infinite sequence of positive natural numbers $n_0 < n_1 < \cdots$ and exactly two digital representations $(a_n)_{n>0}$ and $(b_n)_{n>0}$ of x such that:*

- *$a_k = b_k$, whenever $0 < k < n_0$;*
- *$a_{n_0} = 2$ and $b_{n_0} = 3$;*
- *$a_{n_k} = 5$ and $b_{n_k} = 0$, for odd k;*
- *$a_{n_k} = 0$ and $b_{n_k} = 5$, for even $k > 0$;*
- *$a_i \in \{3, 5\}$ and $a_i - b_i = 3$, whenever $n_{2k} < i < n_{2k+1}$;*
- *$a_i \in \{0, 2\}$ and $b_i - a_i = 3$, whenever $n_{2k+1} < i < n_{2k+2}$.*

Proof According to the chasing algorithm described above in the step n_k the roles of chasing are reversed. But, if the chasing algorithm does not start, then the considered point has a unique digital representation. □

The above theorem makes it easy to check the uniqueness of digital representation. For example, if $x \in \mathbb{X}$ has a digital representation $(x_n)_{n>0}$ such that $x_n = 2$ and $x_{n+1} = 3$ for infinitely many n, then this representation is unique. Indeed, suppose $A = (a_n)_{n>0}$ and $B = (b_n)_{n>0}$ are two different digital representations of a point $x \in \mathbb{X}$ such that $a_k = b_k$, whenever $0 < k < n_0$ and $a_{n_0} = 2 < b_{n_0} = 3$. By Theorem 6.1, the digit 3 never occurs immediately after the digit 2 in digital representations of x for the digits greater than n_0, since it has to be $|a_k - b_k| > 2$ for $k > n_0$.

The map $(a_n)_{n>0} \mapsto \sum_{n>0} \frac{a_n}{4^n}$ is a continuous function from the Cantor set (a homeomorphic copy of the Cantor ternary set) onto the Cantorval such that the preimage of a point has at most two points. In fact, the collection of points with two-point preimages and its complement are both of the cardinality continuum. By the algorithm described above, each sequence $n_0 < n_1 < \cdots$ of positive natural numbers determines exactly two sequences $(a_n)_{n>0}$ and $(b_n)_{n>0}$ such that

$$\sum_{n>0} \frac{a_n}{4^n} = \sum_{n>0} \frac{b_n}{4^n},$$

and vice versa. In the following theorem, we will use the abbreviation $B = \left\{\frac{2}{4^n} : n > 0\right\} \cup \left\{\frac{3}{4^n} : n > 0\right\}$.

Theorem 6.2 *Let $A \subset B$ be such that $B \setminus A$ and A are infinite. Then the set of subsums of a sequence consisting of different elements of A is homeomorphic to the Cantor set.*

Proof Fix a nonempty open interval $\mathbb{I}$. Assume that $(a_n)_{n>0}$ is the digital representation of a point $\sum_{n>0} \frac{a_n}{4^n} \in \mathbb{I}$. Choose natural numbers $m > k$ such that numbers $\sum_{n=1}^{k} \frac{a_n}{4^n}$ and $\frac{5}{4^m} + \sum_{n=1}^{k} \frac{a_n}{4^n}$ belong to $\mathbb{I}$. Then choose $j > m$ such that $\frac{a}{4^j} \in B \setminus A$, where $a = 2$

or $a = 3$. Finally put $b_n = a_n$, for $0 < n \leqslant k$; $b_j = 5$; $b_{j+1} = 2$ and $b_i = 0$ for other cases. Since $b_m = 0$, we get $\sum_{i>0} \frac{b_i}{4^i} \in \mathbb{I}$. Theorem 6.1 together with conditions $b_j = 5$ and $b_{j+1} = 2$ imply that the point $\sum_{i>0} \frac{b_i}{4^i}$ is not in the set of subsums of A. Thus, this set being dense in itself and closed is homeomorphic to the Cantor set. □

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