Title: Hlawka's functional inequality

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Abstract. The paper is devoted to the functional inequality (called by us *Hlawka's functional inequality*)

$$f(x+y) + f(y+z) + f(x+z) \leq f(x+y+z) + f(x) + f(y) + f(z)$$

for the unknown mapping f defined on an Abelian group, on a linear space or on the real line. The study of the foregoing inequality is motivated by Hlawka's inequality:

$$\|x+y\| + \|y+z\| + \|x+z\| \leq \|x+y+z\| + \|x\| + \|y\| + \|z\|,$$

which in particular holds true for all x, y, z from a real or complex inner product space.

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1. Introduction

Let X be a real or complex inner product space and let $x, y, z \in X$ be arbitrary. One can verify the classical identity

$$\|x+y\|^2 + \|y+z\|^2 + \|x+z\|^2 = \|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2. \quad (1)$$

A related inequality

$$\|x+y\| + \|y+z\| + \|x+z\| \leq \|x+y+z\| + \|x\| + \|y\| + \|z\|, \quad (2)$$

which is known as Hlawka's inequality, appeared in 1942 in a paper of Hornich [17].

In 1963 Djoković [9] proved an important generalization of (2), which is often called Hlawka–Djoković inequality; see also Adamović [2,3]. This inequality was obtained independently by Smiley and Smiley [37]. Some additional comments can be found in a paper of Simon and Volkmann [36].

In 1974 Witsenhausen [46] showed the importance of inequality (2) for the geometric properties of normed linear spaces. For a survey of other related results which were known before 1993 the reader is referred to the monograph Mitrinović et al. [26].

More recently, Wada [44] obtained a matrix version of the Hlawka–Djoković inequality. Further extensions are due to Cho et al. [8], Rădulescu and Rădulescu [32] and Honda et al. [15], among others. Integral generalizations of (2) are due to Takahashi, et al. [38], [41], among others. Janous [18] obtained several applications of the Hlawka–Djoković inequality for orthogonal polynomials. Niederreiter and Sloan [29] provided some interesting applications of Hlawka's inequality and in a recent paper of Wu [48] a version of Hlawka's inequality for fuzzy real numbers is given.

Let us also note that inequalities similar to inequality (2) are important in the theory of Aleksandrov spaces. In 2008 Berg and Nikolaev [6, Theorem 6] presented an elegant characterization of $CAT(0)$-spaces using a related inequality. Let us note that an alternative proof of this result was obtained by Sato [35].

A normed linear space for which inequality (2) holds for all x, y, z is called a *Hlawka space* (see e.g. Takahasi et al. [39,40]) or *quadrilateral space* (see Smiley and Smiley [37], Watson [45]). It is easy to provide an example of a Banach space which is not a Hlawka space. It suffices to consider the space $\mathbb{R}^3$ with the supremum norm and to take $x = (1, 1, -1), y = (1, -1, 1)$ and $z = (-1, 1, 1)$. Then

$$\|x + y\| + \|y + z\| + \|x + z\| = 6$$

whereas

$$\|x + y + z\| + \|z\| + \|y\| + \|z\| = 4.$$

Modifying this example we can obtain even more: if x, y, z are the same as before and the space $\mathbb{R}^3$ is equipped with the norm

$$\|(t_1, t_2, t_3)\|_p = (|t_1|^p + |t_2|^p + |t_3|^p)^{\frac{1}{p}},$$

then, after some computations, one can verify that Hlawka's inequality (2) does not hold for every $x, y, z \in X$ for any $p > \log_{1.5} 3 \approx 2.71$ (see Witsenhausen [46]).

Each inner product space is a Hlawka space (see e.g. Mitrinović et al. [26, Chapter XVIII, Section 4]). Moreover, every two dimensional space is a Hlawka space (see Kelly et al. [22]). It is also easy to observe that further examples of Hlawka spaces are L_1 or, more generally, $L_1(X, \mu)$, where (X, μ)

is an arbitrary space with measure. Consequently, by a theorem of Lindenstrauss [24] each two-dimensional real normed linear space E is isomorphically isometric to a subset of $L^1([0,1])$ and therefore E is a Hlawka space (this was independently proved in an elementary way in [22]). Moreover, Witsenhausen [46, Corollary 1.2] showed that the space $L^p(0,1)$ is a Hlawka space for $1 \leq p \leq 2$. Therefore, one can see that all Banach spaces having the property that all its finite dimensional subspaces can be embedded linearly and isometrically in the space $L^p([0,1])$, with some $1 \leq p \leq 2$ are Hlawka spaces (see Niculescu and Persson [28] and Lindenstrauss and Pełczyński [25]). Further, Witsenhausen [47] proved that a finite-dimensional real space with piecewise linear norm is embeddable in L_1 if and only if it is a Hlawka space. However, Neyman [27] showed that in the general case embeddability in L_1 does not characterize Hlawka spaces. Concluding, to the best of the author's knowledge, no characterization of Hlawka spaces is presently known.

It is worth noting that identity (1) does not hold in every Hlawka space. In fact, one can check that (1) implies the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

(put $x = y = u - v, z = 2v$ in (1)), which characterizes inner product spaces among all normed linear spaces (see Fréchet [10] and Jordan and von Neumann [19]).

The present paper is devoted to the functional inequality:

$$f(x+y) + f(y+z) + f(x+z) \leq f(x+y+z) + f(x) + f(y) + f(z), \quad (3)$$

for a real-valued unknown mapping f defined on an Abelian group or on a vector space (Sect. 2) and then, in Sect. 3, on the real line. Let us point out that this functional inequality already appeared in the year 1978 in paper [47] of Witsenhausen.

A related functional equation:

$$f(x+y) + f(y+z) + f(x+z) = f(x+y+z) + f(x) + f(y) + f(z) \quad (4)$$

and also a few more general equations were studied by Kannappan [20] in 1995.

Witsenhausen [47, Lemma 1] proved that each positively homogeneous solution of (3) defined on $\mathbb{R}^n$ is a support function of a centrally symmetric convex body (i.e. of a nonvoid compact convex set). Moreover, solutions of inequality (3) play a significant role in the characterization of zonotopes (see e.g. Witsenhausen [46,47]).

Let us note that several functional inequalities related to inequality (3) have already been discussed by other authors, mainly in connection with subadditivity and convexity. In 1965 Popoviciu [31] provided a characterization of convex mappings defined on an interval I as continuous solutions of the inequality:

$$\sum_{1 \le i_1 < \cdots < i_k \le n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right)$$
$$\le \frac{1}{k}\binom{n-2}{k-2}\left(\frac{n-k}{k-1}\sum_{i=1}^{n} f(x_i) + nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right),$$

where $n \ge 3$ and $2 \le k < n$ are fixed integers and $x_1, \ldots, x_n \in I$ are arbitrary. A particular case of Popoviciu's inequality (with $n = 3$ and $k = 2$) is the inequality:

$$2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{x+z}{2}\right)\right]$$
$$\le 3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z),$$

which plays a significant role in the theory of convex functions (see Niculescu and Persson [28]).

Further results for a system of related inequalities are due to Vasić and Adamović [42], Kečkić [21], Pečarić [30], among others.

Burkill [7] considered the expression:

$$H(f) = f(X \cup Y \cup Z) - f(X \cup Y) - f(X \cup Z) - f(Y \cup Z)$$
$$+ f(X) + f(Y) + f(Z).$$

These studies were developed further by Baston [5], a related result for twice-differentiable real functions was proved by Vasić and Stanković [43].

Our purpose is to contribute to the above-mentioned studies. We will describe all solutions of functional inequality (3) under some additional conditions.

Note that the particular solutions of inequality (3) are $f = \|\cdot\|$ on a Hlawka space and $f = \|\cdot\|^2$ on a Hilbert space. More generally, it is clear that for an arbitrary additive functional $a \colon X \to \mathbb{R}$ and for each additive operator $L \colon X \to Y$ having its values in a Hlawka space or in a Hilbert space, respectively, both mappings

$$X \ni x \mapsto f(x) = \|Lx\| + a(x) \in \mathbb{R}$$

and

$$X \ni x \mapsto f(x) = \|Lx\|^2 + a(x) \in \mathbb{R}$$

satisfy (3). In Sect. 2 we prove the converse statements under some homogeneity assumptions.

2. Inequality (3) on linear spaces

We will deal with functional inequality (3) under some additional homogeneity assumptions. In particular we will provide conditions which are necessary and sufficient for a representation of solutions of inequality (3) as a sum of an additive functional and a norm or a square of a norm of a continuous linear operator.

A subadditive mapping $f\colon X \to \mathbb{R}$ defined on an arbitrary Abelian group X which satisfies the homogeneity condition:

$$f(2x) = 2f(x), \quad x \in X \tag{5}$$

is called *sublinear*. Let us recall a result of Ger [14] which characterizes sublinear mappings on Abelian groups.

Theorem 1 (Ger [14]). *Assume that $(X, +)$ is an Abelian group and $f\colon X \to \mathbb{R}$ is an even and sublinear mapping. Then there exist a Banach space E and an additive mapping $A\colon X \to E$ such that f can be represented in the form:*

$$f(x) = \|A(x)\|, \quad x \in X. \tag{6}$$

We begin with a lemma.

Lemma 1. *Assume that $(X, +)$ is an Abelian group and $f\colon X \to \mathbb{R}$ is arbitrary. If f satisfies functional inequality (3) jointly with $f(0) = 0$, then there exist an additive function $a\colon X \to \mathbb{R}$ and an even function $g\colon X \to \mathbb{R}$ such that $f = a + g$. Moreover, the functional inequality*

$$2g(s + t) - 2g(s - t) \leq g(2s) + g(2t) - g(2s - 2t) \tag{7}$$

is satisfied for all $s, t \in X$.

Proof. Let us define $a\colon X \to \mathbb{R}$ as

$$a(x) = \frac{f(x) - f(-x)}{2}$$

for every $x \in X$. Next, substitute $z = -(x + y)$ in inequality (3) to deduce that

$$a(x + y) \leq a(x) + a(y)$$

for all $x, y \in X$. Therefore, since a is odd, a is additive. Note also that the function $g = f - a$ solves inequality (3) and is even. Next, fix arbitrary $s, t \in X$ and apply (3) for the mapping g with the substitution $(x, y, z) \to (s - t, 2t, s - t)$. We arrive at

$$\begin{aligned}
2g(s + t) + g(2s - 2t) &= g(x + y) + g(y + z) + g(x + z) \\
&\leq g(x + y + z) + g(x) + g(y) + g(z) \\
&= g(2s) + 2g(s - t) + g(2t).
\end{aligned}$$

$\square$

Our next statement generalizes an earlier result of Witsenhausen [47, Lemma 1].

Corollary 1. *Assume that* $(X, +)$ *is an Abelian group and* $f \colon X \to \mathbb{R}$ *fulfils condition* (5). *If* f *satisfies functional inequality* (3), *then there exist a Banach space* E *and additive mappings* $A \colon X \to E$ *and* $a \colon X \to \mathbb{R}$ *such that* f *can be represented in the form:*

$$f(x) = \|A(x)\| + a(x), \quad x \in X. \tag{8}$$

Proof. Clearly, $f(0) = 0$. In view of Lemma 1 and Theorem 1 of Ger the proof will be completed if we prove that each mapping g which solves (7) is sublinear. But this follows immediately from inequality (7) and from the homogeneity condition (5). □

To ensure that each mapping which is of the form (8) solves (3) we need to show that if the group X appearing in Corollary 1 is a Hlawka space, then the space E postulated by Theorem 1 of Ger can be taken as a Hlawka space as well.

Theorem 2. *Assume that* $(X, +)$ *is an Abelian group and* $f \colon X \to \mathbb{R}$ *is arbitrary. Then* f *satisfies functional inequality* (3) *jointly with* (5) *if and only if there exist a Banach space* E, *an additive subgroup* H *of* E *such that inequality* (2) *holds true for all* $x, y, z \in H$ *and additive mappings* $A \colon X \to H$ *and* $a \colon X \to \mathbb{R}$ *such that* f *can be represented in the form* (8).

Proof. The "if" part is obvious.

To prove the "only if" part let us apply Corollary 1 to derive that function f has the representation (8) with some Banach space E. Observe that $H := A(X)$ is an additive subgroup of E. Therefore, to finish the proof we need to check the validity of (2) on H. For arbitrarily fixed $x, y, z \in H$ let us pick $u, v, w \in X$ such that

$$x = A(u), \quad y = A(v), \quad z = A(w).$$

We have

$$\begin{aligned}
&\|x + y\| + \|y + z\| + \|x + z\| \\
&= \|A(u + v)\| + \|A(v + w)\| + \|A(u + w)\| \\
&= f(u + v) + f(v + w) + f(u + w) \\
&\quad - a(u + v) - a(v + w) - a(u + w) \\
&\leq f(u + v + w) + f(u) + f(v) + f(w) \\
&\quad - a(u + v + w) - a(u) - a(v) - a(w) \\
&= \|A(u + v + w)\| + \|A(u)\| + \|A(v)\| + \|A(w)\| \\
&= \|x + y + z\| + \|x\| + \|y\| + \|z\|,
\end{aligned}$$

as claimed. □

If in the foregoing theorem we assume additionally that the domain of f is a Banach space and moreover f is continuous, then we easily see that H is a linear subspace of E and additionally both mappings A and a are continuous.

Corollary 2. *Assume that X is a Banach space and $f\colon X \to \mathbb{R}$ is continuous. Then f satisfies functional inequality* (3) *jointly with $f(2x) = 2f(x)$ for all $x \in X$ if and only if there exist a Hlawka space H, a continuous linear operator $A\colon X \to H$ and a continuous linear functional $a\colon X \to \mathbb{R}$ such that f can be represented in the form* (8).

If X and Y are arbitrary Abelian groups, then a mapping $q\colon X \to Y$ is called *quadratic* if and only if it satisfies the Jordan–von Neumann functional equation:

$$q(x + y) + q(x - y) = 2q(x) + 2q(y)$$

for all $x, y \in X$. It is well known that if the group Y is uniquely divisible by 2, then for each quadratic mapping $q\colon X \to Y$ there exists a biadditive and symmetric mapping $B\colon X \times X \to Y$ such that

$$q(x) = B(x, x)$$

for all $x \in X$ (see Aczél and Dhombres [1, Chapter 11, Proposition 1], compare also with Baron and Volkmann [4, Proposition]). Moreover, if additionally X is a Hilbert space, $Y = \mathbb{R}$ and function q is continuous, then B is a bilinear form and, consequently, there exists a continuous linear operator $L\colon X \to X$ such that q can be represented in the form

$$q(x) = \|Lx\|^2$$

for all $x \in X$.

In the next theorem we will provide a characterization of quadratic mappings via inequality (3).

Theorem 3. *Assume that $(X, +)$ is an Abelian group and $f\colon X \to \mathbb{R}$. Then f satisfies functional inequality* (3) *jointly with*

$$f(2x) = 4f(x), \quad x \in X \tag{9}$$

if and only if f is quadratic. Moreover, if this is the case, then Eq. (4) *is satisfied for all $x, y, z \in X$.*

Proof. First we will prove the "only if" part. Using Lemma 1 and by our assumptions we easily see that the additive mapping a postulated by Lemma 1 vanishes. Next, from (7) we derive the inequality

$$f(s + t) + f(s - t) \leq 2f(s) + 2f(t)$$

for all $s, t \in X$. Fix arbitrary $u, v \in X$ and apply this estimate for $s = u+v, t = u - v$ to obtain

$$4f(u) + 4f(v) = f(2u) + f(2v) \leq 2f(u + v) + 2f(u - v)$$

for all $u, v \in X$. We reached two reverse inequalities and therefore f is quadratic.

To verify the "if" part assume that $f \colon X \to \mathbb{R}$ is a quadratic mapping and fix arbitrary $x, y, z \in X$. It is clear that (9) is satisfied. Let $B \colon X \times X \to \mathbb{R}$ be a biadditive and symmetric mapping such that

$$f(x) = B(x, x), \quad x \in X.$$

Utilizing properties of B one can eventually transform (4) equivalently into an identity. Let us skip the straightforward calculation. $\square$

Corollary 3. *Assume that $(X, +)$ is a Hilbert space and $f \colon X \to \mathbb{R}$ is continuous. Then f satisfies functional inequality (3) jointly with (9) if and only if there exists a continuous linear operator $L \colon X \to X$ such that f can be represented in the form:*

$$f(x) = \|Lx\|^2$$

for all $x \in X$.

We conclude this section with a description of solutions of inequality (3) under a more general homogeneity condition:

$$f(2x) = 3f(x) + f(-x), \quad x \in X. \tag{10}$$

It is clear that this condition is in particular fulfilled by every odd mapping satisfying (5) and by every even mapping satisfying (9).

Theorem 4. *Assume that $(X, +)$ is an Abelian group and $f \colon X \to \mathbb{R}$. Then f satisfies functional inequality (3) jointly with (10) if and only if there exist an additive mapping $a \colon X \to \mathbb{R}$ and a quadratic mapping $q \colon X \to \mathbb{R}$ such that $f = a + q$.*

Proof. The "if" part is obvious.

To prove the "only if" part, let us define mappings $a \colon X \to \mathbb{R}$ and $q \colon X \to \mathbb{R}$ by the formulas

$$a(x) = \frac{f(x) - f(-x)}{2}, \quad q(x) = \frac{f(x) + f(-x)}{2}, \qquad x \in X.$$

By Lemma 1 we get that a is additive. Next, it is easy to see that condition (10) guarantees that q satisfies (9). Moreover, one can check that q fulfils inequality (3). Consequently, by Theorem 3 the mapping q is quadratic. $\square$

Corollary 4. *Assume that $(X, +)$ is a Hilbert space and $f \colon X \to \mathbb{R}$ is continuous. Then f satisfies functional inequality (3) jointly with (10) if and only if there exist a continuous linear operator $L \colon X \to X$ and a continuous linear functional $a \colon X \to \mathbb{R}$ such that f can be represented in the form:*

$$f(x) = \|Lx\|^2 + a(x)$$

for all $x \in X$.

3. Inequality (3) on the real line

In what follows we will deal with solutions of functional inequality (3) on the real line with possibly weak additional assumptions (from now on no homogeneity is imposed upon f). We will be considering the real line $\mathbb{R}$ equipped with the standard Lebesgue measure and we denote by $\overline{\mathbb{R}}$ the set $\mathbb{R} \cup \{-\infty, +\infty\}$. Our main tool in this section are the Dini (extreme unilateral) derivatives.

For an arbitrary mapping $f \colon \mathbb{R} \to \mathbb{R}$ the Dini derivatives are defined as follows:

$$D^{\pm} f(x) = \limsup_{h \to 0\pm} \frac{f(x+h) - f(x)}{h}, \quad D_{\pm} f(x) = \liminf_{h \to 0\pm} \frac{f(x+h) - f(x)}{h}$$

for every $x \in \mathbb{R}$.

It is clear that the Dini derivatives can attain infinite values. Therefore, later in this section each inequality which involves Dini derivatives is to be understood that it is valid provided that both its sides are meaningful (i.e. no indefinite expressions of the form $\infty - \infty$ appears).

Banach proved that if f is measurable, then all Dini derivatives of f are measurable as well (see e.g. Saks [34, Chapter IV.4]).

Let us also recall Denjoy–Young–Saks Theorem (see e.g. Saks [34, Chapter IX.4]).

Theorem 5 (Denjoy–Young–Saks). *Assume that I is an interval and $f \colon I \to \mathbb{R}$ is an arbitrary function. Then there exists a set of measure zero $C \subset I$ such that for all $x \in I \setminus C$ exactly one of the following cases holds true:*

  (i) f is differentiable at x;
 (ii) $D_- f(x) = D^+ f(x)$ is finite, $D^- f(x) = +\infty$ and $D_+ f(x) = -\infty$;
(iii) $D_+ f(x) = D^- f(x)$ is finite, $D^+ f(x) = +\infty$ and $D_- f(x) = -\infty$;
 (iv) $D_- f(x) = D_+ f(x) = -\infty$ and $D^- f(x) = D^+ f(x) = +\infty$.

We will begin the study of functional inequality (3) on the real line with some lemmas.

Lemma 2. *Assume that $f \colon \mathbb{R} \to \mathbb{R}$ satisfies functional inequality (3) for all $x, y, z \in \mathbb{R}$ jointly with $f(0) = 0$. Then:*

$$D_+ f(x) + D^+ f(-x) \leq 2D^+ f(0), \tag{11}$$

$$2D_+ f(0) \leq D_+ f(x) + D^+ f(-x) + D^+ f(0) - D_- f(0), \tag{12}$$

$$2D_- f(0) \leq D^- f(x) + D_- f(-x), \tag{13}$$

$$D^- f(x) + D_- f(-x) + D_- f(0) - D^+ f(0) \leq 2D^- f(0), \tag{14}$$

for all $x \in \mathbb{R}$.

Proof. We will prove (11) and (12) only. Proofs of (13) and (14) can be obtained by a modification of the original reasonings.

Fix arbitrary $x \in \mathbb{R}$ and $y > 0$ and apply (3) with substitution $z = -x$. We see that

$$\frac{f(x+y) - f(x)}{y} + \frac{f(-x+y) - f(-x)}{y} \leq 2\frac{f(y)}{y}. \tag{15}$$

Next, let us pick a sequence $(y_n)_{n \in \mathbb{N}}$ (possibly depending upon x) of positive real numbers which tend to zero and

$$\lim_{n \to +\infty} \frac{f(-x+y_n) - f(-x)}{y_n} = D^+ f(-x).$$

On replacing y_n by its suitable subsequence we may assume additionally that both sequences

$$\frac{f(x+y_n) - f(x)}{y_n}, \quad \frac{f(y_n)}{y_n}$$

are convergent in $\overline{\mathbb{R}}$. Next, apply (15) for $y = y_n$, pass $n \to +\infty$ and estimate the two remaining limits by $D_+ f(x)$ and $2D^+ f(0)$, respectively, to derive inequality (11).

Further, for arbitrarily fixed $x \in \mathbb{R}$ and $y > 0$, apply (3) with substitution $(x, y, z) \to (x+y, -y, -x+y)$. We reach

$$2\frac{f(2y)}{2y} \leq \frac{f(x+y) - f(x)}{y} + \frac{f(-x+y) - f(-x)}{y} + \frac{f(y)}{y} - \frac{f(-y)}{-y}. \tag{16}$$

This time we choose a sequence $(y_n)_{n \in \mathbb{N}}$ of positive real numbers tending to zero such that

$$\lim_{n \to +\infty} \frac{f(x+y_n) - f(x)}{y_n} = D_+ f(x)$$

and the remaining limits are convergent. Applying an analogous reasoning as before for estimate (16) we prove inequality (12). $\qquad\square$

Lemma 3. *Assume that $f \colon \mathbb{R} \to \mathbb{R}$ satisfies functional inequality* (3) *for all $x, y, z \in \mathbb{R}$ jointly with $f(0) = 0$. If $|D^+ f(0)| < \infty$, then the mapping*

$$-D_+ f + D^+ f(0) \colon \mathbb{R} \to \overline{\mathbb{R}}$$

is subadditive; and if $|D_- f(0)| < \infty$, then the mapping

$$D^- f - D_- f(0) \colon \mathbb{R} \to \overline{\mathbb{R}}$$

is subadditive.

Proof. Let us keep $x, z \in \mathbb{R}$ temporarily fixed. For each $y > 0$ we deduce from (3) the following inequality:

$$\frac{f(x+y) - f(x)}{y} + \frac{f(z+y) - f(z)}{y} \leq \frac{f(x+z+y) - f(x+z)}{y} + \frac{f(y)}{y},$$

and its reverse for each $y < 0$.

Now, pick a sequence $(y_n)_{n\in\mathbb{N}}$ of positive real numbers which tend to zero such that

$$\lim_{n\to+\infty}\frac{f(x+z+y_n)-f(x+z)}{y_n}=D_+f(x+z).$$

By passing $y\to 0+$ we obtain

$$D_+f(x)+D_+f(z)\le D_+f(x+z)+D^+f(0). \qquad (17)$$

Analogously, in case $y<0$ we can take a sequence $(y_n)_{n\in\mathbb{N}}$ of positive real numbers tending to zero such that

$$\lim_{n\to+\infty}\frac{f(x+z+y_n)-f(x+z)}{y_n}=D^-f(x+z),$$

to get

$$D^-f(x)+D^-f(z)\ge D^-f(x+z)+D_-f(0). \qquad (18)$$

From estimates (17) and (18) we easily see that if the respective Dini derivative is finite at the origin, then the mapping $D_+f-D^+f(0)$ is superadditive whereas the mapping $D^-f-D_-f(0)$ is subadditive, respectively. $\qquad\square$

Remark 1. It is well known that for an arbitrary function f the equalities $D_\pm f=+\infty$ and $D^\pm f=-\infty$ can hold on a set which is at most countable (see e.g. Saks [34]). Therefore both subadditive mappings spoken of in the foregoing lemma are strictly greater than $-\infty$ outside a countable set. Consequently, if we weaken assumptions of this lemma to: $D^+f(0)<+\infty$ or $D_-f(0)>-\infty$, then the assertion should be replaced by:

   there exists a countable set $C\subset\mathbb{R}$ such that the mapping $-D_+f+D^+f(0)$ is well-defined on $\mathbb{R}\setminus C$ and is subadditive on $\mathbb{R}\setminus C$, or

   then the mapping $D^-f-D_-f(0)$ is well-defined on $\mathbb{R}\setminus C$ and is subadditive on $\mathbb{R}\setminus C$, respectively.

Now, we will recall a useful property of subadditive functions that can attain infinite values, which was first proved by Rosenbaum [33] (see also Hille and Phillips [16, Theorem 7.3.3]).

Theorem 6 (Rosenbaum). *If a subadditive measurable function $\varphi\colon\mathbb{R}\to\overline{\mathbb{R}}$ satisfies $\varphi(x_0)<+\infty$ for some $x_0<0$, then either $\varphi(x)=+\infty$ for almost all $x>0$ or $\varphi<+\infty$ on $\mathbb{R}$.*

From this theorem we deduce the following useful fact.

Proposition 1. *Assume that $\varphi\colon\mathbb{R}\to\overline{\mathbb{R}}$ is a subadditive measurable function. If $\varphi(t)<+\infty$ for some $t<0$ and for some $t>0$, then either:*

(i) $\varphi(x) = +\infty$ *for almost all $x \in \mathbb{R}$; or*
(ii) $\varphi = -\infty$ *on $\mathbb{R}$; or*
(iii) φ *is finite on $\mathbb{R}$.*

Proof. Let us apply Theorem 6 of Rosenbaum twice, for φ and then for φ replaced by a map $\mathbb{R} \ni t \mapsto \varphi(-t)$. We deduce that either $\varphi(x) = +\infty$ for almost all $x \in \mathbb{R}$ or $\varphi < +\infty$ on $\mathbb{R}$. Now, assume that the second possibility holds true and $\varphi(t_0) = -\infty$ for some $t_0 \in \mathbb{R}$. Then for arbitrary $t \in \mathbb{R}$ we have

$$f(t) = f(t - t_0 + t_0) \leq f(t - t_0) + f(t_0) = -\infty.$$

Therefore, we see that either $\varphi = -\infty$ or $\varphi > -\infty$ on $\mathbb{R}$, which is precisely what we need. $\qquad \square$

Lemma 4. *Assume that $f \colon \mathbb{R} \to \mathbb{R}$ is measurable and satisfies functional inequality (3) for all $x, y, z \in \mathbb{R}$ jointly with $f(0) = 0$. If all the Dini derivatives of f are finite at the origin and additionally at least one of the Dini derivatives is finite at a negative point and at a positive point, then either f is differentiable almost everywhere and moreover $D^- f$ and $D_+ f$ are finite everywhere, or $D_- f = D_+ f = -\infty$ and $D^- f = D^+ f = +\infty$ almost everywhere on $\mathbb{R}$.*

Proof. Due to Lemma 2 we get that if $D_\pm f$ is finite at a negative point and at a positive point, then the same is true for $D^\pm f$, and vice versa. Thus, at least one of the subadditive mappings postulated by Lemma 3 fulfils the assumptions of Proposition 1. Therefore, we infer that either:

(a) $D^- f = +\infty$ almost everywhere on $\mathbb{R}$ or $D_+ f = -\infty$ almost everywhere on $\mathbb{R}$;

or

(b) at least one of the mappings $D^- f, D_+ f$ is finite (everywhere).

In case (a), by Lemma 2, either $D_- f = -\infty$ and $D^- f = +\infty$ almost everywhere, or $D_+ f = -\infty$ and $D^+ f = +\infty$ almost everywhere. To finish the proof it is enough to apply the Denjoy–Young–Saks Theorem to obtain equalities $D_\pm f = -\infty$ and $D^\pm f = +\infty$ almost everywhere.

In case (b), by Lemma 2 we see that both mappings $D_+ f$ and $D^+ f$, or $D^- f$ and $D_- f$ are finite everywhere. Next, by the Denjoy–Young–Saks Theorem we deduce that the equalities $D_+ f = D^- f$ and $D_- f = D^+ f$ hold almost everywhere, which gives us that f is differentiable almost everywhere. In particular all Dini derivatives of f are finite at some negative point and at some positive point. Having this, we can repeat the preceding argumentation involving Lemma 3 and Proposition 1 to deduce that both $D^- f$ and $D_+ f$ are finite everywhere, as claimed. $\qquad \square$

Remark 2. In the foregoing proof we did not use the finiteness of one of the Dini derivatives at a positive and at a negative point in its full strength. An

inspection of the proof shows that it is enough to assume that one of the lower derivatives is strictly smaller than $+\infty$ or one of the upper derivatives is strictly greater than $-\infty$ at a positive and at a negative point. The same is true for all our subsequent results which involves assumption $(*)$ below. However, it is well known that each of the equalities $D^{\pm}f = -\infty$ and $D_{\pm}f = +\infty$ can hold only on a countable set. Therefore, the possible usefulness of this observation seems to be limited.

In what follows we will need to impose some additional assumptions upon a mapping $f\colon \mathbb{R} \to \mathbb{R}$:

$(*)$ *The function f is measurable, $f(0) = 0$, all the Dini derivatives of f are finite at the origin, at least one of them is finite at a negative point and at a positive point and at least one of them is finite on a set of positive measure.*

In what follows we will need a special case of a result of Gajda [12, Corollary 3.1]. Let us mention that a similar result was obtained earlier by Ger [13] with constant $3C$ instead of C in formula (20) below.

Theorem 7 (Gajda [12]). *Assume that $g\colon \mathbb{R} \to \mathbb{R}$ and $C \geq 0$ are arbitrary. If the estimate*

$$|g(x+y) - g(x) - g(y)| \leq C \tag{19}$$

is satisfied for almost all $(x,y) \in \mathbb{R} \times \mathbb{R}$, then there exists a unique additive mapping $a\colon \mathbb{R} \to \mathbb{R}$ such that

$$|g(x) - a(x)| \leq C \tag{20}$$

for almost all $x \in \mathbb{R}$.

In the next lemma we prove a uniform approximation almost everywhere of the derivative of f.

Lemma 5. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ fulfils $(*)$. If f satisfies functional inequality (3) for all $x, y, z \in \mathbb{R}$, then f is differentiable almost everywhere. Moreover, there exists a constant $A \in \mathbb{R}$ such that*

$$|f'(x) - 2Ax - B| \leq C \tag{21}$$

for almost all $x \in \mathbb{R}$, where $B = \frac{1}{2}[D^{+}f(0) + D_{-}f(0)]$ and $C = \frac{1}{2}[D^{+}f(0) - D_{-}f(0)]$.

Proof. Lemma 4 and our assumption $(*)$ ensure that f is differentiable almost everywhere. Denote by $K \subset \mathbb{R}$ the set on which f is differentiable and keep $x, y \in K$ temporarily fixed. By Lemma 3 we obtain the inequalities

$$f'(x) + f'(y) \leq f'(x+y) + D^{+}f(0)$$

and

$$f'(x) + f'(y) \geq f'(x+y) + D_{-}f(0).$$

Therefore, we reach the estimate

$$D_-f(0) \leq f'(x) + f'(y) - f'(x+y) \leq D^+f(0). \tag{22}$$

Next, define the mapping $g \colon \mathbb{R} \to \mathbb{R}$ as follows. Put $g(t) = f'(t) - B$ for all $t \in K$, where the constant B is specified in the assertion, and put $g(t) = 0$ for $t \in \mathbb{R} \setminus K$. Then (22) easily implies that inequality (19) is valid for all $(x, y) \in K \times K$, where C is given in the assertion. By Theorem 7 of Gajda we obtain the existence of a unique additive mapping $a \colon \mathbb{R} \to \mathbb{R}$ such that estimate (20) holds true for almost all $x \in \mathbb{R}$. One can see that since g is in particular measurable, then a is bounded on a set of a positive measure and thus it cannot be a discontinuous additive mapping (for a comprehensive study of additive functions on the real line, including the discontinuous ones, see e.g. Kuczma [23]). Therefore, there exists a constant $A \in \mathbb{R}$ such that $a(x) = 2Ax$ for all $x \in \mathbb{R}$. Consequently, we have

$$|g(x) - 2Ax| \leq C$$

for almost all $x \in \mathbb{R}$, which is equivalent to (21). $\square$

The next theorem is the main result of this section.

Theorem 8. *Assume that $f \colon \mathbb{R} \to \mathbb{R}$ fulfils $(*)$. If f satisfies functional inequality (3) for all $x, y, z \in \mathbb{R}$, then there exist a constant $A \in \mathbb{R}$ and a mapping $r \colon \mathbb{R} \to \mathbb{R}$ such that*

$$f(x) = Ax^2 + Bx + r(x)$$

and

$$|r(x)| \leq C|x| \tag{23}$$

for all $x \in \mathbb{R}$, where $B = \frac{1}{2}[D^+f(0) + D_-f(0)]$ and $C = \frac{1}{2}[D^+f(0) - D_-f(0)]$.

Proof. By Lemma 4 we infer that at least two Dini derivatives of f are finite everywhere. Therefore, f is equal to the Henstock integral of one of them (see e.g. Fremlin [11, 483X(m), page 232]). Clearly, f is measurable, so are all its Dini derivatives, and thus the Henstock integral of f' coincides with its Lebesgue integral. Next, from Lemma 5 we deduce the estimate

$$-C \leq f'(x) - 2Ax - B \leq C.$$

Therefore, we get

$$-Cy \leq f(y) - Ay^2 - By = \int_0^y f'(x)\, dx - Ay^2 - By \leq Cy \tag{24}$$

if $y > 0$, together with the reverse inequality for $y < 0$. In order to finish the proof it is enough to define $r \colon \mathbb{R} \to \mathbb{R}$ as

$$r(y) = f(y) - Ay^2 - By$$

for all $y \in \mathbb{R}$. The estimate (23) can be easily derived from inequality (24). $\square$

The following corollary is straightforward.

Corollary 5. *Assume that* $f\colon \mathbb{R} \to \mathbb{R}$ *satisfies* $(*)$ *and* $D^+ f(0) \leq D_- f(0)$. *Then* f *satisfies functional inequality* (3) *for all* $x, y, z \in \mathbb{R}$ *if and only if there exist constants* $A, B \in \mathbb{R}$ *such that* $f(x) = Ax^2 + Bx$ *for all* $x \in \mathbb{R}$.

We will terminate the paper with two easy examples illustrating the necessity of some of our assumptions.

Example 1. Note that each mapping $f\colon \mathbb{R} \to [3, 4]$, smooth or not, solves (3). Therefore, the assumption appearing in most of our statements that $f(0) = 0$ cannot be dropped.

Next, if $a\colon \mathbb{R} \to \mathbb{R}$ is a discontinuous additive function, then both mappings $f(x) = a(x)$ and $f(x) = a(x)x$ for all $x \in \mathbb{R}$ solve (3). Therefore, even in the case $f(0) = 0$ we see that the regularity assumptions we have imposed upon f cannot be omitted as well.

References

[1] Aczél, J., Dhombres, J.: Functional equations in several variables, Encyclopedia of Mathematics and its Applications, vol. 31. Cambridge University Press, Cambridge (1989)

[2] Adamović, D.D.: Généralisation d'une identité de Hlawka et de l'inéglité correspondante. Mat. Vesnik **1**(16), 39–43 (1964)

[3] Adamović, D.D.: Quelques remarques relatives aux g'en'eralisations des in'egalit'es de Hlawka et de Hornich. Mat. Vesnik **1**(16), 241–242 (1964)

[4] Baron, K., Volkmann, P.: On orthogonally additive functions. Publ. Math. Debr. **52** (3–4), 291–297 (1998)

[5] Baston, V.J.: On some Hlawka-type inequalities of Burkill. J. Lond. Math. Soc. II. Ser. **12**, 402–404 (1976)

[6] Berg, I.D., Nikolaev, I.G.: Quasilinearization and curvature of Aleksandrov spaces. Geom. Dedicata **133**, 195–218 (2008)

[7] Burkill, J.C.: The concavity of discrepancies in inequalities of the means and of Hölder. J. Lond. Math. Soc. II. Ser. **7**, 617–626 (1974)

[8] Cho, Y.J., Matić, M., Pečarić, J.E.: Inequalities of Hlawka's type in n-inner product spaces. Commun. Korean Math. Soc. **17**(4), 583–592 (2002)

[9] Djoković, Ž.D.: Generalizations of Hlawka's inequality. Glasnik Mat.-Fiz. Astromon. Ser. II. Društvo Mat. Fiz. Hrvatske **18**, 169–175 (1963)

[10] Fréchet, M.: Sur la définition axiomatique d'une classe d'espaces vectoriels distancies applicables vectoriellement sur l'espace de Hilbert. Ann. Math. (2) **36**, 705–718 (1935)

[11] Fremlin, D.H.: Measure Theory, vol. 4, Part II. Torres Fremlin, Colchester (2003)

[12] Gajda, Z.: Invariant Means and representation of Semigroups in the Theory of Functional Equations, Prace Naukowe Uniwersytetu Śląskiego w Katowicach nr. 1273. Uniwersytet Śląski, Katowice (1992)

[13] Ger, R.: Almost Approximately Additive Mappings, General Inequalities 3, ISNM 64, pp. 263–276. Birkhäuser Verlag, Basel (1983)

[14] Ger, R.: Fischer-Muszély additivity on abelian groups. In: Comment. Math. Prace Mat., Tomus Specialis in honorem Juliani Musielak, pp. 82–96 (2004)

[15] Honda, A., Okazaki, Y., Takahashi, Y.: Generalizations of the Hlawka's inequality. Bull. Kyushu Inst. Technol. Pure Appl. Math. **45**, 9–15 (1998)

[16] Hille, E., Phillips, R.S.: Colloquium publications. In: Functional analysis and semigroups, vol. 31, American Mathematical Society, Providence (1957)

[17] Hornich, H.: Eine Ungleichung für Vektorlängen. Math. Z. **48**, 268–274 (1942)

[18] Janous, W.: The Hlawka–Djoković inequality and points in unitary spaces. Gen. Math. **15**(1), 67–74 (2007)

[19] Jordan, P., Neumann, J.von : On inner products in linear, metric spaces. Ann. Math. (2) **36**, 719–723 (1935)

[20] Kannappan, Pl.: Quadratic functional equation and inner product spaces. Results Math. **27**(3–4), 368–372 (1995)

[21] Kečkić, J.D.: On an infinite system of functional inequalities. Publ. Fac. Électrotech. Univ. Belgrade, Sér. Math. Phys. **274–301**, 79–81 (1969)

[22] Kelly, L.M., Smiley, D.M., Smiley, M.F.: Two dimensional spaces are quadilateral spaces. Am. Math. Mon. **72**, 753–754 (1965)

[23] Kuczma, M.: An introduction to the theory of functional equations and inequalities, 2nd edn. Birkhäuser, Basel (2009)

[24] Lindenstrauss, J.: On the extension of operators with a finite-dimensional range. Ill. J. Math. **8**, 488–499 (1964)

[25] Lindenstrauss, J., Pełczyński, A.: Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications. Studia Math. **29**, 275–326 (1968)

[26] Mitrinović, D.S., Pečarić, J.E., Fink, A.M.: Classical and new inequalities in analysis. Kluwer Academic Publishers, Dordrecht (1993)

[27] Neyman, A.: Representation of L_p-norms and isometric embeddings in L_p-spaces. Isr. J. Math. **48**, 129–138 (1984)

[28] Niculescu, C.P., Persson, L.-E.: Convex functions and their applications. A contemporary approach. In: CMS Books in Mathematics (Ouvrages de Mathématiques de la SMC 23), vol. xvi. Springer, New York, NY (2005)

[29] Niederreiter, H., Sloan, I.H.: Variants of the Koksma–Hlawka inequality for vertex-modified quasi-Monte Carlo integration rules. Math. Comput. Model. **23**(8–9), 69–77 (1996)

[30] Pečarić, J.E.: Modified version of a general result of Vasić-Adamović-Kečkić and some remarks concerning inequalities for convex functions. Glas. Mat. **21**(41), 331–341 (1986)

[31] Popoviciu, T.: Sur certaines inégalités qui caractérisent les fonctions convexes. An. Sti. Univ. Al. I. Cuza Iasi N. Ser. Sect. Ia **11B**, 155–164 (1965)

[32] Rădulescu, M., Rădulescu, S.: Generalizations of Dobrushin's inequalities and applications. J. Math. Anal. Appl. **204**(3), 631–645 (1996)

[33] Rosenbaum, R.A.: Sub-additive functions. Duke Math. J. **17**, 227–247 (1950)

[34] Saks, S.: Theory of the integral, 2nd revised edn. With two addit. notes by S. Banach. G.E. Stechert & Co., New York (1937) [Reprinted by Dover Publications, New York (1964)]

[35] Sato, T.: An alternative proof of Berg and Nikolaev's characterization of $CAT(0)$-spaces via quadrilateral inequality. Arch. Math. **93**(5), 487–490 (2009)

[36] Simon, A., Volkmann, P.: On two geometric inequalities. Ann. Math. Sil. **9**, 137–140 (1995)

[37] Smiley, D.M., Smiley, M.F.: The polygonal inequalities. Am. Math. Mon. **71**, 755–760 (1964)

[38] Takahashi, Y., Wada, S., Takahasi, S.-E.: An extension of Hlawka's inequality. Math. Inequal. Appl. **3**(1), 63–67 (2000)

[39] Takahasi, S.-E., Takahashi, Y., Honda, A.: A new interpretation of Djoković's inequality. J. Nonlinear Convex Anal. **1**(3), 343–350 (2000)

[40] Takahasi, S.-E., Takahashi, Y., Miyajima, S., Takagi, H.: Convex sets and inequalities. J. Inequal. Appl. **2**, 107–117 (2005)

[41] Takahashi, Y., Takahasi, S.-E., Wada, S.: A general Hlawka inequality and its reverse inequality. Math. Inequal. Appl. **12**(1), 1–10 (2009)

[42] Vasić, P.M., Adamović, D.D.: Sur un syste'me infini d'inégalités fonctionnelles. Publ. Inst. Math. Nouv. Sér. **9**(23), 107–114 (1969)

[43] Vasić, P.M., Stanković, L.R.: Some inequalities for convex functions. Math. Balk. **6**, 281–288 (1976)

[44] Wada, S.: Hlawka's inequality for matrices. RIMS Kokyuroku **1189**, 138–143 (2001)

[45] Watson, D.: On quadrilateral spaces. Am. Math. Mon. **74**(10), 1213–1215 (1967)

[46] Witsenhausen, H.S.: Metric inequalities and the zonoid problem. Proc. Am. Math. Soc. **40**, 517–520 (1973)

[47] Witsenhausen, H.S.: A support characterization of zonotopes. Mathematika (London) **25**, 13–16 (1978)

[48] Wu, X.: Inequalities of Hlawka type for fuzzy real numbers. Math. Inequal. Appl. **15**(3), 675–684 (2012)

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