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The Hahn–Banach theorem almost everywhere

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Dedicated to Professor Roman Ger on the occasion of his seventieth birthday

Abstract. The aim of this work is to present an almost everywhere version of the Hahn–Banach extension theorem.

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1. Introduction

In the year 1960 Erdös [3] raised the following problem: suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation

$$f(x + y) = f(x) + f(y),$$

for almost all $(x, y) \in \mathbb{R}^2$ (in the sense of the planar Lebesque measure). Does there exists an additive function $a : \mathbb{R} \to \mathbb{R}$ [i.e. a satisfies $a(x + y) = a(x) + a(y)$, for all $(x, y) \in \mathbb{R}^2$] such that

$$f(x) = a(x)$$

almost everywhere in $\mathbb{R}$ (in the sense of the linear Lebesque measure)? A positive answer to this question was given by de Bruijn [2] (and, independently, by Jurkat [9]). N. G. de Bruijn has put the Erdös problem into a more general setting.

Let $(G, +)$ be a group. A non-empty family $\mathcal{I}$ of subsets of G is called a proper linearly invariant ideal (briefly p.l.i. ideal) iff it satisfies the following conditions

(i) $G \notin \mathcal{I}$;
(ii) if $U \in \mathcal{I}$ and $V \subset U$, then $V \in \mathcal{I}$;
(iii) if $U, V \in \mathcal{I}$, then $U \cup V \in \mathcal{I}$;
(iv) if $U \in \mathcal{I}$ and $x \in G$, then $x - U \in \mathcal{I}$.

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For a p.l.i. ideal $\mathcal{I}$ of subsets of a group G we say that a given condition is satisfied $\mathcal{I}$-*almost everywhere* in G (written $\mathcal{I}$-*a.e.*) iff there exists a set $Z \in \mathcal{I}$ such that this condition is satisfied for every $x \in G \backslash Z$.

The set belonging to the set ideal are regarded as, in a certain sense, small sets (see Kuczma [10]). For example, if G is a second category topological commutative group then the family of all first category subsets of G is a p.l.i. ideal. If G is a commutative locally compact topological group equipped with the Haar measure μ then the family of all subsets of G which have zero measure is a p.l.i. ideal. Moreover, if G is a normed space ($\dim G \geq 1$) then the family of all bounded subsets of G is a p.l.i. ideal and also, if G is a commutative uncountable group then the family of all countable subsets of G is a p.l.i. ideal.

Let $(G, +)$ be a commutative group. For a p.l.i. ideal $\mathcal{I}$ we may define the following family of subsets of $G \times G$ (Ger [5,6]):

$$\Omega(\mathcal{I}) = \{N \subset G \times G : N[x] \in \mathcal{I} \ \ \mathcal{I}\text{-}a.e. \ \text{ in } \ G\},$$

where

$$N[x] = \{y \in G : (x, y) \in N\}$$

a subset N of $G \times G$ belongs to $\Omega(\mathcal{I})$ iff there exists a set $U \in \mathcal{I}$ such that

$$N[x] \in \mathcal{I}, \ \ x \in G \backslash U$$

(an abstract version of the Fubini theorem)]. The family $\Omega(\mathcal{I})$ is a p.l.i. ideal of subsets of $G \times G$.

The de Bruijn result can be formulated as follows:

Theorem 1.1. *If $(G, +)$ and $(H, +)$ are commutative groups, $\mathcal{I}$ is a p.l.i. ideal of subsets of G then for every $\Omega(\mathcal{I})$-almost additive function $f : G \to H$, i.e.*

$$f(x + y) = f(x) + f(y) \quad \Omega(\mathcal{I})\text{-}a.e. \quad \text{in} \quad G \times G,$$

there exists a unique homomorphism $a : G \to H$ such that

$$f(x) = a(x) \quad \mathcal{I}\text{-}a.e. \quad \text{in} \quad G.$$

Ger [7] generalized de Bruijn's theorem to the case of non-commutative groups. The notion of p.l.i. ideals and its properties and applications we can find in [10]. One of the most interesting applications is included in the paper of Ger [8] where the author combines the notions of approximately additive and almost additive mappings.

In this paper we proved the following version of the Hahn–Banach extension theorem.

Theorem 1.2. *Let $(H, +)$ be a subgroup of a commutative group $(G, +)$, let $\mathcal{I}$ be a p.l.i. ideal of subsets of G and let $H \notin \mathcal{I}$. Assume that $p : G \to \mathbb{R}$ satisfies*

$$p(x + y) \leq p(x) + p(y) \quad \Omega(\mathcal{I})\text{-}a.e. \quad \text{in} \quad G \times G. \tag{1.1}$$

Then for every additive function $a : H \to \mathbb{R}$ *fulfilling*

$$a(x) \le p(x) \quad \mathcal{I}\text{-}a.e. \quad \text{in} \quad H \tag{1.2}$$

there exists an additive function $A : G \to \mathbb{R}$ *such that*

$$A(x) = a(x) \quad \mathcal{I}\text{-}a.e. \quad \text{in} \quad H. \tag{1.3}$$

and

$$A(x) \le p(x) \quad \mathcal{I}\text{-}a.e. \quad \text{in} \quad G. \tag{1.4}$$

2. Proof of the theorem

Assume that $\mathcal{I}$ is a p.l.i. ideal of subsets of a commutative group $(G, +)$. For a real function f on G we define $\mathcal{I}_f$ to be the family of all sets $Z \in \mathcal{I}$ such that f is bounded on the complement of Z. A real function f on G is called $\mathcal{I}$-*essentially bounded* if and only if the family $\mathcal{I}_f$ is non-empty. The space of all $\mathcal{I}$-essentially bounded functions on G will be denoted by $B^{\mathcal{I}}(R, \mathbb{R})$.

For every element f of the space $B^{\mathcal{I}}(G, \mathbb{R})$ the real numbers

$$\mathcal{I}\text{-essinf}_{x\in G} f(x) = \sup_{Z\in\mathcal{I}_f} \inf_{x\in G\setminus Z} f(x),$$
$$\mathcal{I}\text{-esssup}_{x\in G} f(x) = \inf_{Z\in\mathcal{I}_f} \sup_{x\in G\setminus Z} f(x)$$

are correctly defined and are referred to as the $\mathcal{I}$-*essential infimum* and the $\mathcal{I}$-*essential supremum* of the function f, respectively.

From the Gajda theorem (Gajda [4], see also Badora [1]) we can derive the following.

Theorem 2.1. *If* $\mathcal{I}$ *is a p.l.i. ideal of subsets of a commutative group* $(G, +)$*, then there exists a real linear functional* $M^{\mathcal{I}}$ *on the space* $B^{\mathcal{I}}(G, \mathbb{R})$ *such that*

$$\mathcal{I}\text{-}essinf_{x\in G} f(x) \le M^{\mathcal{I}}(f) \le \mathcal{I}\text{-}esssup_{x\in G} f(x) \tag{2.1}$$

and

$$M^{\mathcal{I}}({}_z f) = M^{\mathcal{I}}(f), \tag{2.2}$$

for all $f \in B^{\mathcal{I}}(G, \mathbb{R})$ *and all* $z \in G$*, where the function* ${}_z f : G \to \mathbb{R}$ *is defined as follows*

$${}_z f(x) = f(z + x), \quad x \in G.$$

Now we prove our result.

Proof of Theorem 1.2. Notice that if $\mathcal{I}$ is a p.l.i. ideal of subsets of G, then the family

$$\mathcal{I} \cap H = \{Z \cap H : Z \in \mathcal{I}\}$$

is a p.l.i. ideal of subsets of H.

From condition (1.1) we infer the existence of the set $U_1 \in \mathcal{I}$ such that for every $x \in G \backslash U_1$ there exists a set $V_x \in \mathcal{I}$ such that

$$p(x+y) \le p(x) + p(y), \quad g \in G \backslash V_x. \tag{2.3}$$

From (1.2) it follows that there exists a set $U_0 \in \mathcal{I}$ such that

$$a(x) \le p(x), \quad x \in H \backslash U_0. \tag{2.4}$$

Let $U = U_1 \cup (-U_1)$. Next we choose arbitrary $x \in G \backslash U$. By (2.4) and (2.3) we get

$$a(z) \le p(z) = p(x - x + z) \le p(x) + p(-x + z), \tag{2.5}$$

for all $z \in H \backslash (U \cup U_0 \cup (x + V_x))$. From (iv) with $x = 0$ we get $-V_x \in \mathcal{I}$. Using again (iv) we infer that $x + V_x \in \mathcal{I}$. Moreover $U \in \mathcal{I}$. Therefore (2.5) means that the function

$$H \ni z \mapsto a(z) - p(-x + z) \in \mathbb{R}$$

is $\mathcal{I}$-essentially bounded from above. So, we can define the function $\varphi : G \to \mathbb{R}$ as follows

$$\varphi(x) = \begin{cases} \mathcal{I}\text{-esssup}_{z \in H}(a(z) - p(-x+z)), & x \in G \backslash U \\ 0, & x \in U. \end{cases}$$

Let $N = (U \times G) \cup (G \times U) \cup \{(x, y) \in G \times G : x + y \in U\}$. For every $x \in G \backslash U$ we have

$$\begin{aligned} N[x] &= \emptyset \cup U \cup \{y \in G : x + y \in U\} \\ &= U \cup \{y \in G : y \in -x + U\} = U \cup (-x + U) \in \mathcal{I}. \end{aligned}$$

Consequently $N \in \Omega(\mathcal{I})$.

Let $(x, y) \in G \times G \backslash N$ be fixed. Then $x \notin U$, $y \notin U$ and $x + y \notin U$. For $z \in G \backslash ((y + V_{-x}) \cup (x + y + V_x))$, by (2.3), we have

$$p(-x - y + z) \le p(-x) + p(-y + z)$$

and

$$p(-y + z) = p(x - x - y + z) \le p(x) + p(-x - y + z)$$

which leads to the following

$$-p(-x) \le p(-y + z) - p(-x - y + z) \le p(x).$$

From this we get

$$\begin{aligned} \varphi(x + y) &= \mathcal{I}\text{-esssup}_{z \in H}(a(z) - p(-y - x + z)) \\ &= \mathcal{I}\text{-esssup}_{z \in H}(a(z) - p(-y + z) + p(-y + z) - p(-x - y + z)) \\ &\le \varphi(y) + p(x) \end{aligned}$$

and

$$\begin{aligned}\varphi(y) &= \mathcal{I}\text{-esssup}_{z\in H}(a(z) - p(-y+z)) \\ &= \mathcal{I}\text{-esssup}_{z\in H}(a(z) - p(-x-y+z) + p(-x-y+z) - p(-y+z)) \\ &\le \varphi(x+y) + p(-x).\end{aligned}$$

Hence

$$-p(-x) \le \varphi(x+y) - \varphi(y) \le p(x), \quad (x,y) \in G \times G\backslash N. \tag{2.6}$$

The last inequalities imply that, for $x \in G\backslash U$, the function

$$G\backslash(U \cup (-x+U)) \ni y \mapsto \varphi(x+y) - \varphi(y) \in \mathbb{R}$$

is bounded which yields that the function

$$G \ni y \mapsto \varphi(x+y) - \varphi(y) \in \mathbb{R}$$

belongs to the space $B^{\mathcal{I}}(G,\mathbb{R})$.

A function $\alpha : G \to \mathbb{R}$ we define by the formula

$$\alpha(x) = \begin{cases} M_y^{\mathcal{I}}(\varphi(x+y) - \varphi(y)), & x \in G\backslash U \\ 0, & x \in U, \end{cases}$$

where $M^{\mathcal{I}}$ is a linear functional whose existence guarantees Theorem 2.1 and the subscript y indicates that the functional $M^{\mathcal{I}}$ is applied to a function of the variable y.

If we choose $(x,y) \in G \times G\backslash N$ then, by the linearity of $M^{\mathcal{I}}$ and (2.2), we get

$$\begin{aligned}\alpha(x) + \alpha(y) &= M_z^{\mathcal{I}}(\varphi(x+z) - \varphi(z)) + M_z^{\mathcal{I}}(\varphi(y+z) - \varphi(z)) \\ &= M_z^{\mathcal{I}}(\varphi(x+y+z) - \varphi(y+z)) + M_z^{\mathcal{I}}(\varphi(y+z) - \varphi(z)) \\ &= M_z^{\mathcal{I}}(\varphi(x+y+z) - \varphi(z)) = \alpha(x+y).\end{aligned}$$

The function α is $\Omega(\mathcal{I})$-almost additive and from Theorem 1.1 we obtain the existence of an additive function $A : G \to \mathbb{R}$ such that

$$A(x) = \alpha(x) \quad \mathcal{I}\text{-}a.e. \quad \text{in} \quad G. \tag{2.7}$$

Next, let $x \in H\backslash U$ be fixed and let $y \in G\backslash(U \cup (-x+U) \cup N[x])$. Then

$$\begin{aligned}\varphi(x+y) &= \mathcal{I}\text{-esssup}_{z\in H}(a(z) - p(-y-x+z)) \\ &= \inf_{Z\in\mathcal{I}} \sup_{z\in H\backslash Z}(a(z) - p(-y-x+z)) \\ &= \inf_{Z\in\mathcal{I}} \sup_{z\in H\backslash(-x+Z)}(a(x+z) - p(-y+z)) \\ &= \inf_{Z\in\mathcal{I}} \sup_{z\in H\backslash(-x+Z)}(a(x) + a(z) - p(-y+z)) \\ &= a(x) + \varphi(y).\end{aligned}$$

Therefore, for $x \in H\backslash U$, using (2.1) we have

$$\alpha(x) = M_y^{\mathcal{I}}(\varphi(x+y) - \varphi(y)) = M_y^{\mathcal{I}}(a(x)) = a(x),$$

which jointly with (2.7) gives (1.3) . Finally, from the definition of α, (2.1), (2.6) and (2.7) we infer that condition (1.4) is satisfied which ends the proof. □

3. Ending comments

Remark 3.1. Note that we can strengthen Theorem 1.2 assuming that the function a is $\mathcal{I}$-almost additive. Then we start the proof from Theorem 1.1.

Remark 3.2. If, in Theorem 1.2, additionally we assume that the functional p is positively homogeneous then we can prove that the function A is linear.

Indeed, for a fixed $x \in G$ let us observe that condition (1.4) implies that

$$A(tx) \le p(tx) = tp(x), \quad t \in (0, +\infty),$$

which means that the real additive function

$$\mathbb{R} \ni t \mapsto A(tx) \in \mathbb{R}$$

is bounded from above, for example, on the interval $(0, 1)$. Whence this function is linear (continuous), i. e.

$$A(tx) = t \cdot c_x, \quad t \in \mathbb{R},$$

for some constant $c_x \in \mathbb{R}$. Putting $t = 1$ we get $c_x = A(x)$ and

$$A(tx) = tA(x), \quad t \in \mathbb{R}.$$

Hence A is a linear map.

Remark 3.3. We will show that the assumption (iv) imposed on the family $\mathcal{I}$ (symmetry with respect to zero) is essential in our theorem.

Let $G = \mathbb{R}$ and let $H = \mathbb{Z}$. The family $\mathcal{I}$ of subsets of $\mathbb{R}$ we define as follows: $A \in \mathcal{I}$ iff A is a countable subset of the interval $(c, +\infty)$, for some $c \in \mathbb{R}$.

Then $H \notin \mathcal{I}$, the family $\mathcal{I}$ satisfies conditions (i)–(iii) of the definition of a p.l.i. ideal [but the condition (iv) is not fulfilled].

Taking $p : \mathbb{R} \to \mathbb{R}$ as $p(x) =| x |$, for $x \in \mathbb{R}$, and $a : \mathbb{Z} \to \mathbb{Z}$ as $a(x) = 2x$, for $x \in \mathbb{Z}$, we have that p satisfies (1.1) and a, p fulfill (1.2) [because $\mathbb{Z} \cap (0, +\infty) \in \mathcal{I}$].

If $A : \mathbb{R} \to \mathbb{R}$ is an additive function satisfying (1.4), then it is bounded from above on some interval (p is bounded from above on each bounded interval). Therefore A is a linear map. So, $A(x) = cx$, $x \in \mathbb{R}$, for some constant $c \in \mathbb{R}$. Moreover $\mathbb{Z} \notin \mathcal{I}$ and if the function A fulfills (1.3), then $A(x) = 2x$, for $x \in \mathbb{R}$, which is impossible because A, p satisfy (1.4) and $(0, +\infty) \notin \mathcal{I}$.

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