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A characterization of quadratic-multiplicative mappings

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Abstract In the spirit of some earlier studies of Jean Dhombres, Roman Ger and Ludwig Reich we discuss the alienation problem for quadratic and multiplicative mappings.

Keywords Ring homomorphisms · Additive-multiplicative functions · Quadratic-multiplicative functions · Alienation phenomenon

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1 Introduction

Assume that P and R are arbitrary rings and $f: P \to R$. Mapping f is called additive if it satisfies the additive Cauchy functional equation:

$$f(x+y) = f(x) + f(y), \quad x, y \in P.$$

Next, f is called multiplicative if it satisfies the multiplicative Cauchy functional equation:

$$f(xy) = f(x)f(y), \quad x, y \in P.$$

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Further, f is said to be quadratic if it satisfies the Jordan-von Neumann functional equation:

$$f(x+y)+f(x-y)=2f(x)+2f(y), \quad x,y\in P.$$

Mappings which are simultaneously additive and multiplicative (in what follows we will call it briefly additive-multiplicative) are simply ring homomorphisms. One may ask about the connection between quadratic-multiplicative mappings and ring homomorphisms. First result in this direction for real-to-real functions is due to Hammer and Volkmann [7]:

Theorem A (C. Hammer, P. Volkmann) *Assume that f is real-to-real mapping. Then f is quadratic-multiplicative if and only if it can be written as*

$$f(x)=(\Re w(x))^2+(\Im w(x))^2=|w(x)|^2,$$

with additive-multiplicative $w\colon \mathbb{C}\to\mathbb{C}$.

A generalization of this theorem is due to Gajda [2]. If R is a field then we denote by $\overline{R}$ the algebraic closure of R and if $\zeta\in\overline{R}$ then $R(\zeta)$ stands for the smallest field containing $R\cup\{\zeta\}$.

Theorem B (Z. Gajda) *Assume that P is a commutative unitary ring and R is a field with characteristic different from 2. A mapping $f\colon P\to R$ is quadratic-multiplicative if and only if it can be represented in the form*

$$f(x)=u(x)\cdot v(x), \tag{1}$$

where both $u\colon P\to R(\zeta)$ and $v\colon P\to R(\zeta)$ are additive-multiplicative mappings such that

$$u(x)+v(x)\in R, \qquad u(x)-v(x)\in\zeta R, \tag{2}$$

where $\zeta\in\overline{R}$ is an element which satisfies $\zeta^2\in R$.

In 1988 Dhombres [1] studied relations between a system of equations defining ring homomorphisms:

$$\begin{cases} f(x+y)=f(x)+f(y),\\ f(xy)=f(x)f(y), \end{cases} \tag{3}$$

a single functional equation which is a sum of this system:

$$f(x+y)+f(xy)=f(x)+f(y)+f(x)f(y), \tag{4}$$

and a more general equation:

$$af(xy)+bf(x)f(y)+cf(x+y)+d(f(x)+f(y))=0, \tag{5}$$

for mapping f defined on a unitary ring which is divisible by 2 and having values in a field. In particular Dhombres provided sufficient conditions under which each solution of (4) is a ring homomorphism. Later this results has been developed and generalized in a few directions by Ger [3–5], Ger and Reich [6], Laohakosol, Pimsert and Udomkavanich [9], among others. The term "alienation phenomenon" was introduced in order to characterize the effect that each solution of a single equation satisfies the corresponding system of equations.

The purpose of the present paper is to obtain analogues of the above-mentioned results for quadratic-multiplicative mappings defined on a commutative unitary ring uniquely divisible by 2 and having values in a commutative ring with no zero divisors.

2 Main results

Theorem 1 *Assume that P is a commutative unitary ring uniquely divisible by 2, R is a commutative ring with no zero divisors and $a, b, c, d, k \in R$ are arbitrary elements. If a given mapping $f: P \to R$ fulfills the following functional equation*:

$$af(xy) + bf(x)f(y) + cf(x+y) + df(x-y) + k(f(x) + f(y)) = 0, \tag{6}$$

for each $x, y \in P$ then either:

(a) $f = 0$ *on* P; *or*

(b) f *is constant and equal to a non zero solution t of equation*

$$bt = -a - c - d - 2k$$

over the ring R; *or*

(c) $d = 0$ *and* $c = -k - bf(0)$; *or*

(d) $c = d \neq 0, k = -2c - bf(0)$ *and* f *is even.*

Proof If f is constant and equal to some $t \in R$ then obviously $t = 0$ or, since R has no zero divisors,

$$a + bt + c + d + 2k = 0.$$

Assume that f is nonconstant on P. First, let us consider the case where $f(0) = 0$.

If $d = 0$ then (6) reduces to (5) and we may apply a result of Dhombres [1, Thm 11] which implies that $c = -k$.

Therefore, let us assume that $d \neq 0$. Replace x by y and simultaneously y by x in (6) to get

$$af(yx) + bf(y)f(x) + cf(y+x) + df(y-x) + k(f(y) + f(x)) = 0$$

for each $x, y \in P$. Compare this with (6) and use the commutativity of P and R to arrive at:

$$df(x-y) = df(y-x)$$

for each $x, y \in P$. Therefore, since R contains no zero divisors, then $f(x-y) = f(y-x)$ for each $x, y \in P$, i.e. f is even.

Set $x = 0$ in (6) to obtain

$$cf(y) + df(-y) + kf(y) = 0$$

for every $y \in P$. Since $f \neq 0$ then we deduce the equality $c + d + k = 0$.

Put $-y$ instead of y in (6) and use the evenness of f to see that

$$af(xy) + bf(x)f(y) + cf(x-y) + df(x+y) + k(f(x) + f(y)) = 0,$$

for each $x, y \in P$. Compare this with (6) to derive the equality

$$(c-d)[f(x+y) - f(x-y)] = 0, \tag{7}$$

for each $x, y \in P$. Now, since f is nonconstant we may take $u, v \in P$ such that $f(u) \neq f(v)$. Next, make use of the fact that P is uniquely divisible by 2 and apply (7) with substitutions $x = (u+v)/2$, $y = (u-v)/2$ to arrive at

$$(c-d)\,[f(u) - f(v)] = (c-d)\left[f\left(\frac{u+v}{2} + \frac{u-v}{2}\right) - f\left(\frac{u+v}{2} - \frac{u-v}{2}\right)\right] = 0.$$

From this equality and from the fact that R contains no zero divisors we deduce that $c - d = 0$.

Next, we will drop the assumption $f(0) = 0$. To do this substitute $g(x) := f(x) - f(0)$ in (6) to see that

$$ag(xy) + bg(x)g(y) + cg(x+y) + dg(x-y) + k'(g(x) + g(y)) + A = 0,$$

for each $x, y \in P$, where

$$k' = k + bf(0)$$

and

$$A = af(0) + bf(0)^2 + cf(0) + df(0) + 2kf(0).$$

On the other hand, it is straightforward to check that $A = 0$ (substitute $x = y = 0$ in (6)). Therefore, we may apply the previous reasonings for the map g and for constants a, b, c, d and k' to derive assertions postulated in points (c) and (d).

The case (c) in the foregoing theorem has been thoroughly discussed in the literature (see Introduction). Therefore, in what follows we will confine ourselves to the case (d). Moreover, as it was shown in the last part of the proof of Theorem 1, we may additionally assume without loss of generality that $f(0) = 0$. In this situation, Eq. (6) reduces to:

$$af(xy) + bf(x)f(y) + cf(x + y) + cf(x - y) - 2c(f(x) + f(y)) = 0, \quad (8)$$

for each $x, y \in P$ and $c \neq 0$.

In our next step, we embed ring R in its field of fractions. Therefore, we may think of Eq. (8) as an equation over the field. This is made with the purpose of facilitate some calculations in the target space and the necessary conditions on the solutions hold also true after the embedding. Hence all our consecutive statements have consequences also for solutions with values in a ring.

First let us deal with the case $a \neq 0$ and $b \neq 0$. Replace f by $-ab^{-1}f$ and multiply the resulted equation by $-a^{-2}b$ to arrive at

$$f(xy) - f(x)f(y) = \omega[2f(x) + 2f(y) - f(x + y) - f(x - y)], \quad (9)$$

for each $x, y \in P$, where $\omega = -a^{-1}c$.

Theorem 2 *Assume that P is a commutative unitary ring uniquely divisible by* 2 *and divisible by* 3*, R is a field and $\omega \in R$ is an arbitrary nonzero element. If a given mapping $f: P \to R$ such that $f(0) = 0$ fulfills Eq.* (9) *for each $x, y \in P$ then f is a quadratic-multiplicative mapping.*

Proof By Theorem 1 f is even (a direct proof of this fact is also possible: it suffices to interchange the roles of variables x, y in (9) and subtract the resulted equation from (9)).

In what follows we are going to apply the technique of "duplication" formulas and "triplication" formulas, which already has been applied several times to analogous problems in previous papers on the topic, e.g. Ger [3,4], Ger and Reich [6].

For the sake of brevity let us denote $\alpha = f(1)$ and $\beta = f(2)$. Substitute $y = 1$ in (9) to obtain

$$f(x + 1) + f(x - 1) = \gamma f(x) + 2\alpha, \quad (10)$$

for every $x \in P$, where $\gamma = 2 - (1 - \alpha)\omega^{-1}$. Next, put $y = 2$ in (9) to deduce the equality

$$f(2x) = (2\omega + \beta)f(x) + 2\omega\beta - \omega[f(x + 2) + f(x - 2)], \quad (11)$$

for every $x \in P$. This jointly with (10) leads to the following "duplication formulae":

$$\begin{aligned} f(2x) &= (4\omega + \beta)f(x) + 2\omega\beta - \omega[f(x + 2) + f(x) + f(x) + f(x - 2)] \\ &= (4\omega + \beta)f(x) + 2\omega\beta - \omega[\gamma f(x + 1) + 2\alpha + \gamma f(x - 1) + 2\alpha] \\ &= (4\omega + \beta)f(x) + 2\omega(\beta - 2\alpha) - \omega\gamma[\gamma f(x) + 2\alpha] \\ &= \lambda f(x) + \chi, \end{aligned}$$

for every $x \in P$, where $\lambda = 4\omega + \beta - \omega\gamma^2$ and $\chi = 2\omega(\beta - 2\alpha - \gamma\alpha)$. Note that since $f(0) = 0$ then $\chi = 0$. Consequently, $\beta = (2 + \gamma)\alpha$ and the foregoing formula reduces to

$$f(2x) = \lambda f(x) \tag{12}$$

for every $x \in P$. Apply this for $x = 1$ to see that $\beta = \lambda\alpha$, which implies that either $\alpha = \beta = 0$ or $\lambda = 2 + \gamma$. But if $\alpha = \beta = 0$ then one may check that we have $\gamma = 2 - \omega^{-1}$ and also

$$\lambda = (4 - (2 - \omega^{-1})^2)\omega = 4 - \omega^{-1} = 2 + \gamma.$$

This equality after some straightforward calculations leads to the following identity:

$$(\beta - 4\alpha)\omega = \alpha(\alpha - 1). \tag{13}$$

Let us firstly deal with the case $\lambda = 0$. Clearly $f(2x) = 0$ for every $x \in P$. Since P is uniquely divisible by 2 then $f = 0$ on P.

Now, let us consider the case $\lambda = 1$. Replace y by $2y$ in (9) to deduce the equality:

$$\begin{aligned}\omega[2f(x) + 2f(y) - f(x + 2y) - f(x - 2y)]\\ = f(2xy) - f(x)f(2y) = f(xy) - f(x)f(y)\\ = \omega[2f(x) + 2f(y) - f(x + y) - f(x - y)],\end{aligned}$$

for each $x, y \in P$. Applying the facts that f is even and $\lambda = 1$ we get the following equation:

$$f(x + 2y) + f(x - 2y) = f(x + y) + f(x - y),$$

for each $x, y \in P$. Put $y = x$ to see that $f(3x) = 0$ for every $x \in P$. Taking the advantage of the fact that P is divisible by 3 we eventually deduce the equality $f = 0$ on P.

Now, assume that $\lambda \neq 0$ and $\lambda \neq 1$. Substitute $x := 2x$ and $y := 2y$ in (9) and apply "duplication formulae" (12) to obtain

$$\lambda^2 f(xy) - \lambda^2 f(x)f(y) = \omega\lambda[2f(x) + 2f(y) - f(x + y) - f(x - y)],$$

for each $x, y \in P$. Next, compare this equality with (9) to arrive at

$$f(xy) - f(x)f(y) = 0, \tag{14}$$

for each $x, y \in P$, i.e. f is multiplicative. This immediately implies that f is also a quadratic mapping and proves point (a) of the assertion.

Let us note the following corollary, which is a consequence of our Theorem 1 and Theorem B of Z. Gajda.

Corollary 1 *Assume that P is a commutative unitary ring uniquely divisible by 2 and divisible by 3, R is a field with characteristic 0 and $\omega \in R$ is an arbitrary nonzero element. A given mapping $f\colon P \to R$ such that $f(0) = 0$ fulfills Eq. (9) for each $x, y \in P$ if and only if there exist a number $\zeta \in \overline{R}$ such that $\zeta^2 \in R$, additive-multiplicative mappings $u\colon P \to R(\zeta)$ and $v\colon P \to R(\zeta)$ such that equalities (1) and (2) hold true.*

Next, let us deal with the case $a \neq 0$ and $b = 0$ in Eq. (8). Multiplication of both sides of this equation by a^{-1} gives us

$$f(xy) = \omega[2f(x) + 2f(y) - f(x+y) - f(x-y)], \tag{15}$$

for each $x, y \in P$, with $\omega = a^{-1}c$.

Theorem 3 *Assume that P is a commutative unitary ring uniquely divisible by 2 and divisible by 3, R is a field and $\omega \in R$ is an arbitrary nonzero element. If a given mapping $f\colon P \to R$ such that $f(0) = 0$ fulfills Eq. (15) for each $x, y \in P$ then $f = 0$.*

Proof Apply Theorem 1 to deduce that f is even (or alternatively, it is enough to interchange the roles of variables x, y in (15) and subtract the resulted equation from (15)).

Next, denote $\alpha = f(1)$ and $\beta = f(2)$ and put $y = 1$ in (15) to get the equality

$$f(x+1) + f(x-1) = \frac{2\omega - 1}{\omega} f(x) + 2\alpha, \tag{16}$$

for every $x \in P$. Next, substitute $y = 2$ in (15) and apply (16) to obtain the following "duplication formulae":

$$\begin{aligned} f(2x) &= \omega[2f(x) + 2\beta - f(x+2) - f(x-2)] \\ &= 4\omega f(x) + 2\omega\beta - \omega[f(x+2) + f(x) + f(x) + f(x-2)] \\ &= 4\omega f(x) + 2\omega\beta - 4\omega\alpha - (2\omega - 1)[f(x+1) + f(x-1)] \\ &= 4\omega f(x) + 2\omega\beta - 4\omega\alpha - (2\omega - 1)\left[\frac{2\omega - 1}{\omega} f(x) + 2\alpha\right] \\ &= \lambda f(x) + \chi, \end{aligned}$$

for every $x \in P$, where $\lambda = 4 - \omega^{-1}$ and $\chi = 2\omega(\beta - 4\alpha) + 2\alpha$. Note that since $f(0) = 0$ then $\chi = 0$. Consequently, $\alpha = \omega(4\alpha - \beta)$ and the foregoing formula reduces to

$$f(2x) = \lambda f(x) \tag{17}$$

for every $x \in P$. Replace x by $2x$ and y by $2y$ in (15). By the "duplication formulae" (17) we deduce the equality

$$\begin{aligned}\lambda^2 f(xy) &= f(4xy) = \omega[2f(2x) + 2f(2y) - f(2x+2y) - f(2x-2y)] \\ &= \lambda\omega[2f(x) + 2f(y) - f(x+y) - f(x-y)] \\ &= \lambda f(xy).\end{aligned}$$

for each $x, y \in P$. Therefore, $f = 0$ unless $\lambda^2 = \lambda \neq 0$. But the latter possibility means that $\lambda = 1$ and we may mimic a part of the previous proof. Replace y by $2y$ in (15) to obtain:

$$\begin{aligned}&\omega[2f(x) + 2f(y) - f(x+2y) - f(x-2y)] \\ &\quad = f(2xy) = f(xy) = f(x)f(y)z \\ &\quad = \omega[2f(x) + 2f(y) - f(x+y) - f(x-y)],\end{aligned}$$

for each $x, y \in P$, which gives us that

$$f(x+2y) + f(x-2y) = f(x+y) + f(x-y),$$

for each $x, y \in P$. Put $y = x$ to see that $f(3x) = 0$ for every $x \in P$ and due to the fact that P is divisible by 3 we obtain $f = 0$ on P, as claimed.

Last interesting for us case of (8) is $a = 0$ and $b \neq 0$. Here we may replace f by $b^{-1}cf$ and multiply the resulted equation by bc^{-2} to get the following functional equation:

$$f(x)f(y) = 2f(x) + 2f(y) - f(x+y) - f(x-y), \tag{18}$$

for each $x, y \in P$.

Theorem 4 *Assume that P is a commutative unitary ring and R is a field with characteristic different from 2. If a given map $f : P \to R$ such that $f(0) = 0$ fulfills Eq. (18) for each $x, y \in P$ then a mapping $g : P \to R$ given by $g(x) = 1 - \frac{1}{2}f(x)$ for every $x \in P$ solves the d'Alembert functional equation*

$$2g(x)g(y) = g(x+y) + g(x-y), \tag{19}$$

for each $x, y \in P$.

Proof Define the function $g : P \to R$ as in the statement of this theorem. A straightforward calculation shows that Eq. (18) after inserting $f = 2 - 2g$ is transformed into:

$$4g(x)g(y) = 2g(x+y) + 2g(x-y), \tag{20}$$

for each $x, y \in P$.

We may apply the foregoing theorem together with a result of Pl. Kannappan [8] describing solutions of the d'Alembert functional equation. He proved that if $(G, +)$ is a group (not necessary Abelian), R is a field with characteristic different from 2 and $g\colon G \to R$ is a solution of the d'Alembert functional Eq. (19) then there exists an exponential mapping $m\colon G \to R$, i.e. a solution of the exponential Cauchy functional equation:

$$m(x+y) = m(x)m(y), \quad x, y \in G,$$

such that

$$g(x) = \frac{m(x) + m(-x)}{2}, \quad x \in G.$$

Therefore, we obtain the following corollary.

Corollary 2 *Assume that P is a commutative unitary ring and R is a field with characteristic different from 2. If a given mapping $f\colon P \to R$ such that $f(0) = 0$ fulfills Eq. (18) for each $x, y \in P$ then there exists an exponential mapping $m\colon P \to R$ such that*

$$f(x) = 2 - m(x) - m(-x), \quad x \in P.$$

Now, let us summarize Theorems 1, 2, 3 and 4 in a single corollary, which is in a sense complementary to a result of Dhombres [1, Thm 11].

Corollary 3 *Assume that P is a commutative unitary ring uniquely divisible by 2 and divisible by 3, R is a field with characteristic different from 2 and $a, b, c, d, k \in R$ are arbitrary elements such that $d \neq 0$. If a given mapping $f\colon P \to R$ fulfills Eq.* (6) *for each $x, y \in P$ then*:

(i) *if $a \neq 0$ and $b \neq 0$ then $c = d, k = -2d - bf(0)$ and the mapping $q = -\frac{b}{a} f + \frac{b}{a} f(0)$ is quadratic-multiplicative*;
(ii) *if $a \neq 0$ and $b = 0$ then $c = d, k = -2d$ and f is constant and equal to zero if $a \neq 2d$ or to an arbitrary constant if $a = 2d$*;
(iii) *if $a = 0$ and $b \neq 0$ then $c = d, k = -2d - bf(0)$ and the mapping $g = -\frac{b}{2d} f + 1 + \frac{b}{2d} f(0)$ solves the d'Alembert functional equation*;
(iv) *if $a = 0$ and $b = 0$ then $c = d, k = -2d$ and the mapping f is quadratic.*

Conversely, each of the mappings described in points (i)–(iv) *above provides a solution of Eq.* (6).

Remark 1 It may be tempting to solve the following equation:

$$af(xy) + bf(x)f(y) + cf(x+y) + df(x-y) + kf(x) + lf(y) = 0, \qquad (21)$$

which is more general than (6), with arbitrary $a, b, c, d, k, l \in R$. This functional equation provide a joint generalization of (6) ($k = l$) and of a general equation discussed the paper [6] ($d = 0$). However, we will show that one cannot expected an elegant behavior of (21), which is characteristic for Eq. (5). From the results quoted in the Sect. 1 and concerning Eq. (5) it follows that for each nontrivial solution f of (5) the following implication holds true: if f solves the equation then automatically the values of the coefficients ab, c, d are uniquely determined (or uniquely up to a multiplicative constant). To see this it not the case for (21) pick arbitrary $\alpha \in R$ and take $a = \alpha, b = -1, c = 1, d = 3, k = -4$ and $l = 2$ and then $a = \alpha, b = -1, c = 1, d = 0, k = l = -1$. We see that in both situations the map $f(x) = \alpha x$ solves (21).

On the other hand, it is possible to provide additional assumptions which imposed upon f will force that either $d = 0$ or $k = l$. For example, it is enough to assume that $f: P \to R$ is a solution of (21) and there exists a $y_0 \in P$ such that $f(y_0) = f(-y_0) \neq 0$. Indeed, it suffices to substitute in (21) $x = y_0$ and $y = 0$ and then $x = 0$ and $y = y_0$ to reach equalities

$$cf(y_0) + df(y_0) + kf(y_0) = 0,$$
$$cf(y_0) + df(-y_0) + lf(y_0) = 0.$$

Since $f(y_0) \neq 0$ and we keep assuming that R contains no zero divisors then $k = l$ and thus we reduced (21) to (6).

References

1. Dhombres, J.: Relations de dépendance entre les équations fonctionnelles de Cauchy. Aequationes Math. **35**, 186–212 (1988)
2. Gajda, Z.: On multiplicative solutions of the parallelogram functional equation. Abh. Math. Sem. Univ. Hamburg **63**, 59–66 (1993)
3. Ger, R.: On an equation of ring homomorphisms. Publ. Math. Debrecen **52**, 397–412 (1998)
4. Ger, R.: Ring homomorphisms equations revisited. Rocznik Naukowo-Dydaktyczny Akademii Pedagogicznej w Krakowie. Prace Matematyczne **17**, 101–115 (2000)
5. Ger, R.: Additivity and exponentiality are alien each to the other. Aequationes Math. (to appear)
6. Ger, R., Reich, L.: A generalized ring homomorphisms equation. Monatsh. Math. **159**, 225–233 (2010)
7. Hammer, C., Volkmann, P.: Die multiplikativen Lösungen der Parallelogrammgleichung. Abh. Math. Sem. Univ. Hamburg **61**, 197–201 (1991)
8. Kannappan, Pl.: The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups. Proc. Amer. Math. Soc. **19**, 69–74 (1968)
9. Laohakosol, V., Pimsert, W., Udomkavanich, P.: Dependence relations among solutions of a universal Cauchy's functional equation. East–West J. Math. Special Vol., pp. 99–108 (2008)